EHRHART QUASIPOLYNOMIALS OF COXETER PERMUTAHEDRA

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CERTIFICATION OF APPROVAL

I certify that I have read *EHRHART QUASIPOLYNOMIALS OF COXETER PERMUTAHEDRA* by Jodi McWhirter and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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The Ehrhart polynomial counts lattice points in a dilated lattice polytope. The Ehrhart polynomials of permutahedra of types A, B, C, and D have been calculated by Federico Ardila, Federico Castillo, and Michael Henley (2015). However, when a type B permutahedron is shifted so that its center is the origin, it becomes a half-integral polytope, and its Ehrhart quasipolynomial was previously unknown. The same is true of odd-dimension type A permutahedra. We use signed graphs that arise from the generating vectors of each permutahedron to determine which sets of vectors are linearly independent and thus which form parallelepipeds that are a part of a zonotopal decomposition, as well as which of these parallelepipeds stays on the lattice when the permutahedron is shifted. This yields new approaches/formulas for Ehrhart quasipolynomials for these rational permutahedra.

I certify that the Abstract is a correct representation of the content of this thesis.

Matthias Beck 5/19/2019

Chair, Thesis Committee Date

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Chapter 1

Introduction

Have you ever found yourself counting something random, like steps in a staircase or beeps of a timer before someone turns it off? It might seem rather random to count the points with integer coordinates, sometimes called **lattice points**, in a figure, but it turns out that many counting problems reduce to counting integer points in, that is, finding the **discrete volume** of, a convex polytope, defined in Section 2.1.

This study of counting integer points in dilations of polytopes is called Ehrhart Theory; Eugène Ehrhart showed in [3] (1962) that this counting function is a polynomial with some quite nice properties if your shape is a convex polytope with integral vertices and a quasipolynomial (defined in Section 2.3) with similar nice properties if your shape is a convex polytope with rational vertices; these theorems appear in Section 2.4. We call these the Ehrhart polynomial and Ehrhart quasipolynomial, respectively.

Though we know some things about Ehrhart polynomials and, to a lesser extent,

Ehrhart quasipolynomials, there is still a lot to discover; we decided to focus on a particular type of convex polytope: the zonotope, defined fully in Section 2.5. A zonotope is a projection of a cube, and it has some more nice properties which we will explore in Section 2.5. One of the properties that makes it particularly convenient for Ehrhart Theory is that we can decompose a zonotope into half-open parallelepipeds; Figure 1 shows such a decomposition for a 2-dimensional type B permutahedron, a zonotope explored in Section 3.1. This decomposition tiles the

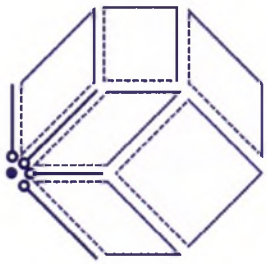

Figure 1.1: A decomposition of an octagon into half-open parallelepipeds.

zonotope, which means that if we want to determine the integer point count of the whole shape, we need only count the integer points in each of the parallelepipeds and add them all up. Richard Stanley in [7] (1997) has a theorem that shows how we can compute the Ehrhart polynomial of an integral zonotope, that is, a zonotope with integral vertices.

How much harder can it be to determine the Ehrhart quasipolynomial of a *rational* zonotope? The answer, unfortunately, is *significantly more difficult.* We look at a few examples of rational cubes in Section 2.6 and compute their Ehrhart quasipoly-

nomials before returning to another type of integral zonotope. In Section 2.7, we meet permutahedra of types A, B, C, and D, whose Ehrhart polynomials are computed in [1]. We take a brief detour in Section 2.8 to introduce signed graphs, an important tool for our study of permutahedra.

In Chapter 3, we translate each type of permutahedra (A, B, C, and D) so that their center is at the origin; this translation ties the permutahedra to their other definition, which comes from taking permutations of some numbers. The C_d-permutahedron, for instance, is the convex hull of signed permutations of $(1, 2, \ldots, d)$. We examine the translated permutahedra, and if the translated versions do not have integral vertices, we compute their Ehrhart quasipolynomial. In Section 3.1, we learn that the type B permutahedra has half integral vertices when translated. Using signed graphs, we learn about the parallelepipeds that make up its zonotopal decompositions and compute its Ehrhart quasipolynomial in Theorem 3.6.

Theorem 3.6. *Let*

$$\Gamma_d := \{\textit{signed graphs on } [d] \textit{ with only CC, HC, and TC}\}$$

and

$$\widetilde{\Gamma_d} := \{\textit{signed graphs on } [d] \textit{ with only CC, HC, and even TC}\}.$$

$\widetilde{\mathcal{Z}}(B_d)$, *the* B_d*-permutahedron centered at the origin, is a half-integral zonotope, and*

$$L_{\widetilde{\mathcal{Z}}(B_d)}(t) = \begin{cases} \sum_{G \in \Gamma_d} \left(2^{cc(G)}\right) t^{d-tc(G)} & \textit{if } t \textit{ even,} \\ \sum_{G \in \widetilde{\Gamma_d}} \left(2^{cc(G)}\right) t^{d-tc(G)} & \textit{if } t \textit{ odd.} \end{cases}$$

In Section 3.2, we discover that some of the type A permutahedra have integral vertices when translated but that others do not, and we apply the same tools that we developed for type B to compute the Ehrhart quasipolynomial for type A in Theorem 3.10.

Theorem 3.10. *Let*

$$F_d := \{\textit{forests on } [d]\},$$

$$\widetilde{F_d} := \{\textit{forests on } [d] \textit{ with only even } TC\}.$$

The permutahedron $\widetilde{\mathcal{Z}}(A_{d-1})$ is a half-integral zonotope, and

$$L_{\widetilde{\mathcal{Z}}(A_{d-1})}(t) = \begin{cases} \sum_{G \in F_d} t^{d-tc(G)} & \textit{if } t \textit{ even,} \\ \sum_{G \in \widetilde{F_d}} t^{d-tc(G)} & \textit{if } t \textit{ odd.} \end{cases}$$

In Sections 3.3 and 3.4, we see that types C and D retain integral vertices when translated.

Chapter 2

Background

2.1 Starting Definitions

For our journey into the realm of counting, we shall consider a class of figures called convex polytopes. A **convex polytope** $\mathcal{P}$ is the convex hull of a finite set of points $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^d$ [2, p. 27]:

$$\mathcal{P} = \{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n : \lambda_1 + \cdots + \lambda_n = 1, \text{ all } \lambda_j \geq 0\}.$$

We will first consider the case in which all of the vertices of the polytope, a collection of the $\mathbf{v}_j$s, have integer coordinates; we will then consider what happens when these points are allowed to be rational. The next question, then, is, what happens when we change the size of our favorite polytope $\mathcal{P}$? In particular, what if we dilate $\mathcal{P}$ by some positive integer t? This is the **lattice-point enumerator** of $t\mathcal{P}$ [2, p. 29],

which can be written as

$$L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right).$$

We shall explore this counting function more thoroughly later; for now, let us also add the character of the generating function of $L_{\mathcal{P}}(t)$, called its **Ehrhart series** [2, p. 30]:

$$\mathrm{Ehr}_{\mathcal{P}}(z) := 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t.$$

2.2 A Square and a Cube: First Examples

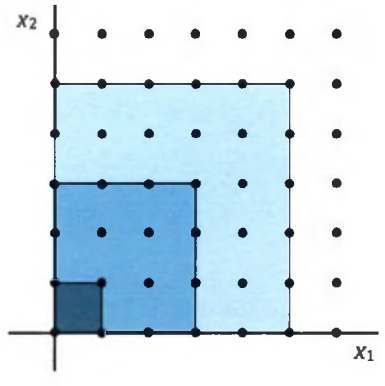

Figure 2.1: Some dilates of $\square_2$.

Consider the unit square, $\square_2 = [0,1]^2$; its discrete volume is 4. One can see that each of the vertices is an integer point but that there are no other integer points in this square. What happens, then, if we dilate our square? Pictured in Figure 2.1 are the original square as well as its 3rd and 5th dilates. One can see

that $L_{\square_2}(t) = (t+1)^2$.

Rather conveniently, we can now consider $\square_d = [0,1]^d$ and observe that $L_{\square_d}(t) = (t+1)^d$. The Ehrhart series, then, is

$$\begin{aligned}\mathrm{Ehr}(z) &= 1 + \sum_{t\geq 1} L_{\square_d}(t) z^t \\ &= 1 + \sum_{t\geq 1} (t+1)^d z^t.\end{aligned}$$

We can rewrite the Ehrhart series in a different way; to do so, we need to introduce the Eulerian numbers. The **Eulerian numbers** $A(d,k)$ are defined in [2, p. 30] by

$$\sum_{j\geq 0} j^d z^j = \frac{\sum_{k=0}^{d} A(d,k) z^k}{(1-z)^{d+1}}.$$

Using the Eulerian numbers, we now have the following theorem from [2]:

Theorem 2.1. *The Ehrhart series of* $\square_d$ *is* $\mathrm{Ehr}_{\square_d}(z) = \frac{\sum_{k=1}^{d} A(d,k) z^{k-1}}{(1-z)^{d+1}}$.

Proof.

$$\begin{aligned}
\mathrm{Ehr}_{\square_d}(z) &= 1 + \sum_{t\geq 1}(t+1)^d z^t \\
&= \sum_{t\geq 0}(t+1)^d z^t \\
&= \frac{1}{z}\sum_{t\geq 1} t^d z^t \\
&= \frac{1}{z}\,\frac{\sum_{k=1}^{d} A(d,k)z^k}{(1-z)^{d+1}} \\
&= \frac{\sum_{k=1}^{d} A(d,k)z^{k-1}}{(1-z)^{d+1}}.
\end{aligned}$$

□

We shall also show that an explicit formula [2] for the Eulerian numbers is

$$A(d,k) = \sum_{j=0}^{k}(-1)^j\binom{d+1}{j}(k-j)^d.$$

Proof. Let $d \in \mathbb{Z}_{\geq 0}$; we have

$$\sum_{j\geq 0} j^d z^j = \frac{\sum_{k=0}^{d} A(d,k)z^k}{(1-z)^{d+1}}.$$

We are interested in the coefficient of z^k in the numerator of the right-hand side. Multiplying both sides of the equation by $(1-z)^{d+1}$ gives

$$(1-z)^{d+1}\sum_{j\geq 0} j^d z^j = \sum_{k=0}^{d} A(d,k)z^k.$$

To find the coefficient of z^k, we add together each part of the product on the left-hand side that results in z^k; however, $(1-z)^{d+1}$ is not currently written in a convenient form for us to do this, but we can rewrite it:

$$\begin{aligned}(1-z)^{d+1} &= (-1)^0\binom{d+1}{0}z^0 + (-1)^1\binom{d+1}{1}z^1 + \cdots + (-1)^{d+1}\binom{d+1}{d+1}z^{d+1} \\ &= \sum_{j=0}^{d+1}(-1)^j\binom{d+1}{j}z^j.\end{aligned}$$

Now, we multiply the coefficient of the z^j in $\sum_{j=0}^{d+1}(-1)^j\binom{d+1}{j}z^j$ by the coefficient of z^{k-j} in $\sum_{j\geq 0} j^d z^j$:

$$A(d,k) = \sum_{j=0}^{k}\left((-1)^j\binom{d+1}{j}\right)\left((k-j)^d\right),$$

which is exactly what we set out to prove. □

2.3 Polygons

Consider a convex polygon $\mathcal{P} \in \mathbb{R}^2$; what can we say about these figures? Pick's Theorem, named after Georg Alexander Pick [4], relates the lattice points of a polygon to its area.

Theorem 2.2 (Pick's Theorem, [4]). *Let $\mathcal{P}$ be a convex polygon with integer vertices; A, the area of $\mathcal{P}$; I, the number of lattice points in the interior of $\mathcal{P}$; and B, the*

number of lattice points on the boundary of $\mathcal{P}$. Then

$$A = I + \frac{1}{2}B - 1.$$

Rather than prove Pick's Theorem, we shall verify that it holds for a rectangle with integer coordinates. Note that, without loss of generality, we can assume that our rectangle $\mathcal{R}$ is in the first quadrant with one vertex on the origin; if it is not in that location initially, we can easily shift its coordinates by integer values, thus not changing the discrete volume, until it is in our desired location. The vertices of this rectangle, then, are $(0,0), (a,0), (0,b)$, and (a,b) for some positive integers a and b.

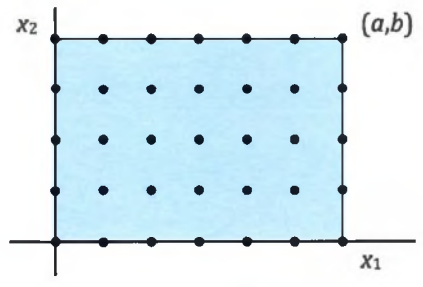

Figure 2.2: A rectangle $\mathcal{R}$ with $a = 6,\ b = 4$.

We now compute A, I, and B of the rectangle $\mathcal{R}$:

$$\begin{aligned} A &= ab \\ I &= (a-1)(b-1) \\ B &= 2(a+1) + 2(b+1) - 4. \end{aligned}$$

Then,

$$\begin{aligned} I + \frac{1}{2}B - 1 &= (a-1)(b-1) + \frac{1}{2}\Big(2(a+1) + 2(b+1) - 4\Big) - 1 \\ &= ab - a - b + 1 + a + 1 + b + 1 - 2 - 1 \\ &= ab \\ &= A. \end{aligned}$$

Lemma 2.3. *Let B and $\mathcal{P}$ be defined as in Theorem 2.2. The number of points on the boundary of $t\mathcal{P}$ is tB.*

Proof. Consider the boundary of $\mathcal{P}$; say $\mathcal{P}$ has n edges. The boundary can be decomposed into n half-open line segments, one for each edge, as shown in Figure 2.3. We can label these half-open edges $e_1, \ldots, e_n$, and say that the number of integer points on each e_k is b_k. Then, since there is no overlapping of half-open edges, $B = \sum_{k=1}^{n} b_k$. It suffices to show, then, that the number of integer points on te_k is tb_k.

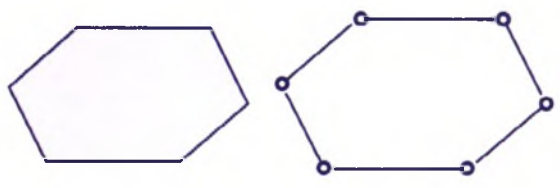

Figure 2.3: A hexagon with its boundary decomposed into half-open edges.

Suppose e_k has endpoints (a, b) and (c, d) so that $e_k = [(a, b), (c, d))$. Then

$te_k = [(ta, tb), (tc, td))$. The slope of e_k is $\frac{d-b}{c-a} = \frac{y}{x}$ for some relatively prime x, y. Note that the only integer points on e_k will occur at points of the form $(a+jx, b+jy)$.

If $d-b$ and $c-a$ are relatively prime, then (a, b) will be the only integer point on e_k, since in that case $(a+x, b+y) = (c, d)$.

Suppose, on the other hand, that $d-b$ and $c-a$ are not relatively prime. Then we decompose e_k into copies of $[(a, b), (a+x, b+y))$. Since each copy of $[(a, b), (a+x, b+y))$ contains exactly one integer point, there must be b_k copies needed.

We can similarly decompose te_k into t copies of e_k, thus giving the number of integer points on te_k to be tb_k.

Thus the number of points on the boundary of $t\mathcal{P}$ is

$$\sum_{k-1}^{n} tb_k = t\sum_{k=1}^{n} b_k = tB. \qquad \square$$

Theorem 2.4. *Let $\mathcal{P}$ be a convex polygon with integer vertices; A, the area of $\mathcal{P}$; and B, the number of lattice points on the boundary of $\mathcal{P}$. Then*

$$L_{\mathcal{P}}(t) = At^2 + \frac{1}{2}Bt + 1.$$

Proof. Let I be the number of lattice points in the interior of $\mathcal{P}$. Rearranging the equation in Pick's Theorem 2.2 gives us that the number of points with integer

coordinates in a polygon $\mathcal{P}$ is

$$I + B = A - \frac{1}{2}B + 1 + B = A + \frac{1}{2}B + 1.$$

Since the area of $t\mathcal{P}$ is At^2 and the number of lattice points on the boundary of $t\mathcal{P}$ is tB, we now have the lattice-point enumerator of $\mathcal{P}$:

$$L_{\mathcal{P}}(t) = At^2 + \frac{1}{2}Bt + 1. \qquad \square$$

One particularly exciting item of note from this theorem is that for an integral polygon $\mathcal{P}$, $L_{\mathcal{P}}(t)$ is a polynomial of degree 2. The lattice-point enumerator of a polygon is not always a polynomial. What happens, for instance, if instead of having only integer coordinates, the vertices of $\mathcal{P}$ have rational coordinates, such as the rectangle in Figure 2.4? No longer is $L_{\mathcal{P}}(t)$ a polynomial; it is a quasipolynomial. A **quasipolynomial** Q is a function of the form

$$Q(t) = \begin{cases} p_0(t) & \text{if } t \equiv 0 \bmod k, \\ p_1(t) & \text{if } t \equiv 1 \bmod k, \\ \vdots & \\ p_{k-1}(t) & \text{if } t \equiv k-1 \bmod k, \end{cases}$$

for some polynomials $p_0, p_1, \ldots, p_{k-1}$ and some positive integer k. The minimal

choice of k is the **period** of Q [2, p. 47].

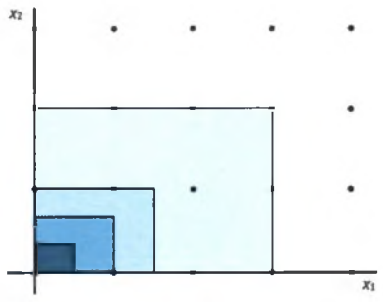

Figure 2.4: The first, second, third, and sixth dilates of a rational rectangle.

Theorem 2.5 ([3]). *Let $\mathcal{P}$ be a convex polygon with rational coordinates. Then $L_{\mathcal{P}}(t)$ is a quasipolynomial of degree 2 whose leading coefficient is the area of $\mathcal{P}$.*

2.4 Cones and Ehrhart Theory

A useful tool in studying convex polytopes is coning over a polytope. Choose a convex d-polytope $\mathcal{P}$ and place this polytope in $\mathbb{R}^{d+1}$ by setting the $(d+1)$st coordinate of each vertex to 1: given vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of $\mathcal{P}$, our new vertices are $(\mathbf{v}_1, 1), (\mathbf{v}_2, 1), \ldots, (\mathbf{v}_n, 1)$ [2, p. 63]. The **cone over** $\mathcal{P}$, then, is

$$\text{cone}(\mathcal{P}) = \{\lambda_1(\mathbf{v}_1, 1) + \lambda_2(\mathbf{v}_2, 1) + \cdots + \lambda_n(\mathbf{v}_n, 1) : \text{ all } \lambda_j \geq 0\}.$$

Taking all the points in this cone that have a $(d+1)$st coordinate of 1 returns a copy of our original polytope, and taking all the points in this cone that have a $(d+1)$st

coordinate of t returns a copy of $t\mathcal{P}$.

We can do more with these cones than cone over polytopes, though we will return to this; we can also list all the integer points contained in a cone. For a set S, let

$$\sigma_S(\mathbf{z}) := \sum_{\mathbf{m} \in S \cap \mathbf{Z}^d} \mathbf{z}^{\mathbf{m}},$$

the **integer-point transform** of S.

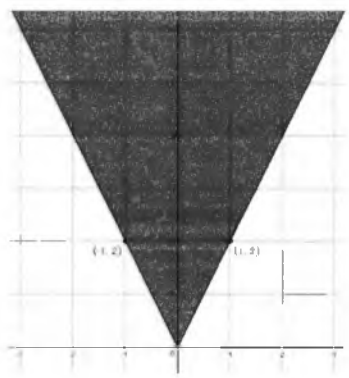

Figure 2.5: The cone with generators $(1, 2)$ and $(-1, 2)$.

Example 2.1. Consider the 2-dimensional cone $\mathcal{K} = \{\lambda_1(1,2)+\lambda_2(-1,2) : \lambda_1, \lambda_2 \geq 0\}$, pictured in Figure 2.5. We say that the **fundamental parallelepiped** of $\mathcal{K}$ is the half-open parallelepiped

$$\Pi := \{\lambda_1(1,2) + \lambda_2(-1,2) : 0 \leq \lambda_1, \lambda_2 < 1\},$$

determined by the first lattice point on each generating ray of $\mathcal{K}$, which we can use

to tile $\mathcal{K}$ and recover each integer point in exactly one copy of Π. Thus there are two things we need to do: first, list all the integer points in a single tile, and second, list all the tiles of $\mathcal{K}$. To do the second, we can use a geometric series:

$$\sum_{j\geq 0}\sum_{k\geq 0} \mathbf{z}^{j(1,2)+k(-1,2)} = \frac{1}{(1-z_1z_2^2)(1-z_1^{-1}z_2^2)}.$$

Next, we determine where the integer points in Π are; in our case, there are 4 integer points in Π: (0,0), (0,1), (0,2), and (0,3). These can be encoded by the polynomial $1+z_2+z_2^2+z_2^3$. Note that setting $(z_1, z_2) = (1, 1)$ in the previous statement yields the number of integer points in Π. Putting these two parts together through multiplication gives

$$\sigma_{\mathcal{K}}(\mathbf{z}) = \frac{1+z_2+z_2^2+z_2^3}{(1-z_1z_2^2)(1-z_1^{-1}z_2^2)}.$$

This construct of $\sigma_{\mathcal{K}}(\mathbf{z})$ is general; that is, the numerator of $\sigma_{\mathcal{K}}(\mathbf{z})$ is the polynomial that encodes the integer points of the fundamental parallelepiped, and the denominator of $\sigma_{\mathcal{K}}(\mathbf{z})$ is the denominator of the geometric series that lists all copies of the fundamental parallelepiped of $\mathcal{K}$ and is constructed from the generators of $\mathcal{K}$.

We are now ready for Ehrhart's Theorem.

Theorem 2.6 (Ehrhart's Theorem, [3]). *Let $\mathcal{P}$ be an integral convex d-polytope; then $L_{\mathcal{P}}(t)$ is a polynomial in t of degree d.*

We have already seen this result for $d = 2$ when we looked at the polygons; now,

we have a result that holds for a general dimension d. In honor of Ehrhart's work, we call $L_{\mathcal{P}}(t)$ the Ehrhart polynomial of $\mathcal{P}$ [2, p. 68]. As with $d = 2$, Ehrhart has a theorem for rational polytopes.

Theorem 2.7 (Ehrhart's Theorem for rational polytopes, [3]). *Let $\mathcal{P}$ be a rational convex d-polytope; then $L_{\mathcal{P}}(t)$ is a quasipolynomial in t of degree d. The period of $L_{\mathcal{P}}(t)$ divides the least common multiple of the denominators of the coordinates of the vertices of $\mathcal{P}$.*

Returning to the idea of coning over a polytope $\mathcal{P}$, recall that the tth dilate of $\mathcal{P}$ can be recovered by taking all the points in cone($\mathcal{P}$) whose $(d+1)$st coordinate is t. For this reason, looking at z_{d+1}^t in $\sigma_{\text{cone}(\mathcal{P})}$ will provide a list of integer points whose $(d+1)$st coordinate is t. Thus

$$\begin{aligned}\sigma_{\text{cone}(\mathcal{P})}(z_1, z_2, \ldots, z_{d+1}) &= 1 + \sigma_{\mathcal{P}}(z_1, \ldots, z_d) z_{d+1} + \sigma_{2\mathcal{P}}(z_1, \ldots, z_d) z_{d+1}^2 \\ &\quad + \sigma_{3\mathcal{P}}(z_1, \ldots, z_d) z_{d+1}^3 + \ldots \\ &= 1 + \sum_{t \geq 1} \sigma_{t\mathcal{P}}(z_1, \ldots, z_d) z_{d+1}^t.\end{aligned}$$

This looks very similar to the Ehrhart series, except that the integer-point transform of $\mathcal{P}$ contains d variables; however, we noted earlier that setting each variable equal to 1 gave the number of lattice points for a particular figure. This holds in general, since each lattice point has exactly one term in the sum that is the

integer-point enumerator. Thus, as can be seen in [2, p. 70],

$$\sigma_{\mathrm{cone}(\mathcal{P})}(1,1,\ldots,1,z_{d+1}) = 1 + \sum_{t\geq 1} \sigma_{t\mathcal{P}}(1,1,\ldots,1) z_{d+1}^t = 1 + \sum_{t\geq 1} L_{\mathcal{P}}(t) z_{d+1}^t.$$

There are many fascinating and useful results that stem from Ehrhart's theorems, a number of which are detailed in [2]. One such result is Stanley's Nonnegativity Theorem.

Theorem 2.8 (Stanley's Nonnegativity Theorem, [6]). *Suppose $\mathcal{P}$ is an integral convex d-polytope with Ehrhart series*

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_0^*}{(1-z)^{d+1}}.$$

Then $h_0^*, h_1^*, \ldots, h_d^*$ are nonnegative integers.

Another result connects the discrete volume of a polytope to its continuous volume [2, p. 77].

Theorem 2.9 ([3]). *Suppose $\mathcal{P} \in \mathbb{R}^d$ is an integral convex d-dimensional polytope with Ehrhart polynomial*

$$c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

Then $c_d = \mathrm{vol}\ \mathcal{P}$.

2.5 Zonotopes

We are now one definition away from meeting the main character of our story: the zonotope. The definition we need is that of a Minkowski sum. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n \subset \mathbb{R}^d$ be polytopes; their **Minkowski sum** is [2, p. 167]

$$\mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_n := \{\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n : \mathbf{x}_j \in \mathcal{P}_j\}.$$

A zonotope is the Minkowski sum of line segments; more formally, given n line segments, each with one endpoint at the origin and the other at $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathbb{R}^d$, the **zonotope**

$$\begin{aligned}\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) &:= \{\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n : \mathbf{x}_j = \lambda_j \mathbf{u}_j \text{ with } \lambda_j \in [0,1]\} + \mathbf{b} \\ &= \{\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \cdots + \lambda_n \mathbf{u}_n : 0 \le \lambda_j \le 1\} + \mathbf{b} \\ &= \mathbf{A}[0,1]^n + \mathbf{b},\end{aligned}$$

where $\mathbf{A}$ is the matrix whose columns are $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ and $\mathbf{b} \in \mathbb{R}^d$.

Figures 2.6 and 2.7 show specific examples of zonotopes, but the previous examples in this paper were also rather conveniently chosen to be examples of zonotopes. The parallelepipeds we came across when looking at the cones are a type of zonotope that will come up again as a particularly useful structure.

Every face of a zonotope occurs when some number of $\lambda_1, \ldots, \lambda_n$ are fixed at 0

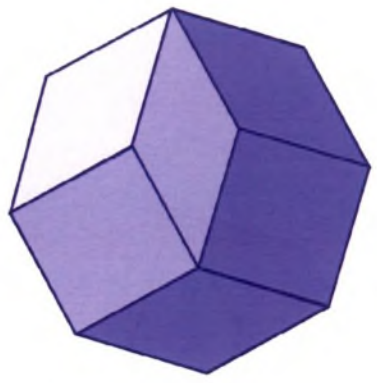

Figure 2.6: A rhombic dodecahedron is a zonotope in $\mathbb{R}^3$ with 4 generators.

or 1. One can show that the face created when some λ_js are fixed at 0 and other λ_is are fixed at 1 and the face created when those same λ_js and λ_is are fixed at 1 and 0, respectively, have the same structure. Thus, if we want to study the structure of some face, it suffices to set all the fixed λ_js to 0. What happens? Let us suppose, for simplicity, that $\lambda_1 = \cdots = \lambda_k = 0$ and $\lambda_{k+1}, \ldots, \lambda_n$ are allowed to vary. That gives us

$$\begin{aligned}\{0\mathbf{u}_1 + \cdots + 0\mathbf{u}_k + \lambda_{k+1}\mathbf{u}_{k+1} + \cdots + \lambda_n\mathbf{u}_n : 0 \leq \mathbf{u}_j \leq 1\}\\ =&\{\lambda_{k+1}\mathbf{u}_{k+1} + \cdots + \lambda_n\mathbf{u}_n : 0 \leq \mathbf{u}_j \leq 1\}\\ =&\mathbf{A}'[0,1]^n,\end{aligned}$$

where $\mathbf{A}'$ is the matrix whose columns are $\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n$; that is:

Proposition 2.10. *Every face of a zonotope is itself a zonotope.*

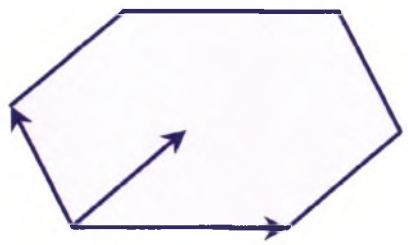

Figure 2.7: A hexagon is a zonotope in $\mathbb{R}^2$ with 3 generators.

One rather useful thing to do with a zonotope is to translate it so that its center of mass is at the origin. To achieve this, we consider

$$\begin{aligned}
2\mathbf{A}[0,1]^n - (\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n) &= \{2\lambda_1\mathbf{u}_1 + \cdots + 2\lambda_n\mathbf{u}_n - (\mathbf{u}_1 + \cdots + \mathbf{u}_n) : 0 \leq \lambda_j \leq 1\} \\
&= \{(2\lambda_1 - 1)\mathbf{u}_1 + \cdots + (2\lambda_n - 1)\mathbf{u}_n : 0 \leq \lambda_j \leq 1\} \\
&= \{\mu_1\mathbf{u}_1 + \cdots + \mu_n\mathbf{u}_n : -1 \leq \mu_j \leq 1\} \\
&= \mathbf{A}[-1,1]^n.
\end{aligned}$$

We denote this dilated and translated zonotope $\mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n)$. One way in which this translated zonotope is useful is that it allows one to see a certain type of symmetry: **symmetry about the origin**. That is, if $\mathbf{x}$ is in our zonotope, then $-\mathbf{x}$ is also in our zonotope. The argument follows fairly quickly from this translation: suppose we have some $\mathbf{x} \in \mathcal{Z}(\pm\mathbf{u}_1, \pm\mathbf{u}_2, \ldots, \pm\mathbf{u}_n)$. Then

$$\mathbf{x} = \lambda_1\mathbf{u}_1 + \cdots + \lambda_n\mathbf{u}_n$$

for some $\lambda_1, \ldots, \lambda_n \in [-1, 1]$. Since $\lambda_j \in [-1, 1]$, it follows that $-\lambda_j \in [-1, 1]$, and

$$-\mathbf{x} = -(\lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n) = -\lambda_1 \mathbf{u}_1 - \cdots - \lambda_n \mathbf{u}_n$$

is also in our zonotope. In general, a zonotope is **centrally symmetric**, which means that some translate of the zonotope is symmetric about the origin.

Now that we have been introduced to the zonotope and know a little about it, we are ready to, quite literally, start picking it apart: every zonotope can be decomposed into half-open parallelepipeds. A formal definition and statement of this result are next, and after those, we will examine a particular zonotope to convince ourselves of the truth of the previous statement.

This definition requires linearly independent vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{R}^d$ and $\sigma_1, \ldots, \sigma_m \in \{\pm 1\}$ [2]. Then

$$\Pi_{\mathbf{w}_1,\ldots,\mathbf{w}_m}^{\sigma_1,\ldots,\sigma_m} := \left\{ \lambda_1 \mathbf{w}_1 + \cdots + \lambda_m \mathbf{w}_m : \begin{array}{l} 0 \leq \lambda_j < 1 \text{ if } \sigma_j = -1 \\ 0 < \lambda_j \leq 1 \text{ if } \sigma_j = 1 \end{array} \right\}.$$

Since a parallelepiped is the Minkowski sum of linearly independent vectors, we can see, with some examination, that $\Pi_{\mathbf{w}_1,\ldots,\mathbf{w}_m}^{\sigma_1,\ldots,\sigma_m}$ is a half-open parallelepiped whose generators are $\mathbf{w}_1, \ldots, \mathbf{w}_m$. With this definition, we can formalize the zonotopal decomposition into half-open parallelepipeds earlier claimed [2, p. 171].

Theorem 2.11 ([5]). *The zonotope $\mathcal{Z}(\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n)$ can be written as a disjoint*

union of translates of $\Pi^{\sigma_1,\ldots,\sigma_m}_{w_1,\ldots,w_m}$, *where* $\{w_1,\ldots,w_m\}$ *ranges over all linearly independent subsets of* $\{u_1,\ldots,u_n\}$, *each equipped with an appropriate choice of signs* $\sigma_1,\ldots,\sigma_m$.

Consider, for example, the hexagon pictured in Figure 2.7. We can label the generators to give them an ordering and make them easier to reference: let the horizontal vector be $\mathbf{u}_1$, the middle vector be $\mathbf{u}_2$, and the vector pointing to the left be $\mathbf{u}_3$. We shall proceed with our decomposition of $\mathcal{Z}(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3)$ starting with $\mathbf{u}_1$.

By itself, $\mathbf{u}_1$ is just a line segment. Thus its decomposition into parallelepipeds consists of a point and a half-open line segment, pictured in Figure 2.8. Specifically, the decomposition we will use is $\mathbf{0}\cup(\mathbf{0},\mathbf{u}_1]$.

Figure 2.8: A decomposition of $\mathcal{Z}(\mathbf{u}_1)$ into half-open parallelepipeds.

In adding $\mathbf{u}_2$, we get a new dimension. We can no longer be content with just half-open line segments and points. We retain the decomposition we used for $\mathbf{u}_1$ and apply the same concept to $\mathbf{u}_2$ without doubling the origin. However, we also need to consider the part of the zonotope that comes from λ_1 and λ_2 both being nonzero. We thus get the parallelepiped generated by $\mathbf{u}_1$ and $\mathbf{u}_2$ but leave the sections that we have already covered open. That is, this parallelepiped is $\Pi^{1,1}_{\mathbf{u}_1,\mathbf{u}_2}=\{\lambda_1\mathbf{u}_1+\lambda_2\mathbf{u}_2 : 0<\lambda_1,\lambda_2\leq 1\}$. Figure 2.9 displays the decomposition given by $\mathbf{u}_1$ with $\mathbf{u}_2$.

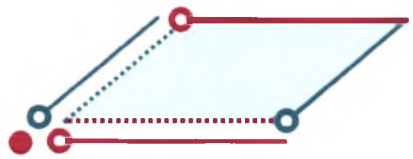

Figure 2.9: A decomposition of $\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2)$ into half-open parallelepipeds.

The addition of $\mathbf{u}_3$ does not add another dimension, like that of $\mathbf{u}_2$ did, so we now have to be careful to only choose linearly independent subsets of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. In particular, we cannot select all three vectors at once. We also do not need to consider $\mathbf{u}_1$ with $\mathbf{u}_2$, since that step was previously completed. Our new half-open parallelepipeds, then, are $(\mathbf{0}, \mathbf{u}_3]$, $\Pi^{1,1}_{\mathbf{u}_1,\mathbf{u}_3} + \mathbf{u}_2 = \{\lambda_1\mathbf{u}_1 + \lambda_3\mathbf{u}_3 + \mathbf{u}_2 : 0 < \lambda_1, \lambda_3 \leq 1\}$ (a translate of the zonotope with generators $\mathbf{u}_1$ and $\mathbf{u}_3$), and $\Pi^{1,1}_{\mathbf{u}_2,\mathbf{u}_3} = \{\lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 : 0 < \lambda_2, \lambda_3 \leq 1\}$. Figure 2.10 displays the completed zonotopal decomposition of Figure 2.7.

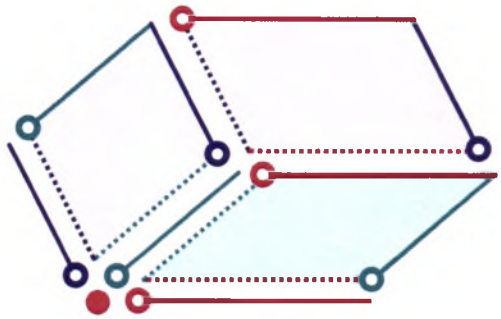

Figure 2.10: A decomposition of $\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ into half-open parallelepipeds.

Figure 2.11 shows a zonotopal decomposition of a polygon we will introduce in Section 2.7 and explore more in Section 3.1.

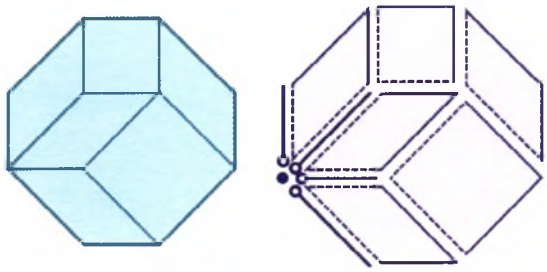

Figure 2.11: A zonotopal decomposition of $\mathcal{Z}(B_2)$.

One of the beautiful things about this decomposition is that it partitions our zonotope, so every integer point that is in our zonotope is in exactly one of the half-open parallelepipeds. Thus, in order to determine the Ehrhart polynomial and Ehrhart series of a zonotope, we might be able to make some substantial progress by looking at the Ehrhart polynomial of a half-open parallelepiped. We shall first consider a d-dimensional parallelepiped in $\mathbb{Z}^d$ [2, p. 172].

Lemma 2.12. *Suppose $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_d \in \mathbb{Z}^d$ are linearly independent, and let*

$$\Pi := \{\lambda_1 \boldsymbol{w}_1 + \lambda_2 \boldsymbol{w}_2 + \cdots + \lambda_d \boldsymbol{w}_d : 0 \leq \lambda_j < 1\}.$$

Then

$$\#(\Pi \cap \mathbb{Z}^d) = \text{vol } \Pi = |\det(\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_d)|,$$

and for every positive integer t,

$$\#(t\Pi \cap \mathbb{Z}^d) = (\text{vol } \Pi)t^d.$$

This lemma uses the fact that integer dilates of a half-open parallelepiped can be tiled by copies of the original half-open parallelepiped—the tth dilate requires exactly t^d such copies—and the fact that the leading term of the Ehrhart polynomial of a polytope is the volume of the polytope, stated earlier in Theorem 2.9.

The following theorem suggests that examining half-open parallelepipeds is exactly what we want to do [2, p. 172].

Theorem 2.13. *Decompose the zonotope $\mathcal{Z} \in \mathbb{R}^d$ into half-open parallelepipeds. Then the coefficient c_k of the Ehrhart polynomial*

$$L_{\mathcal{Z}}(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$$

equals the sum of the relative volumes of the k-dimensional parallelepipeds in the decomposition of $\mathcal{Z}$.

If we have a k-dimensional parallelepiped in $\mathbb{R}^d$ with $k < d$, the volume of this parallelepiped is 0; however, Theorem 2.13 suggests that we can get more information out of something called the *relative volume.* For instance, if we look back at the zonotopal decomposition of the hexagon in Figure 2.10, all the half-open 2-dimensional parallelepipeds will survive with a non-zero volume, but the half-open line segments and point at the origin will come up as 0.

Definition 2.1. Let $S \in \mathbb{R}^d$ be of dimension $k < d$ and span $S = \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in S, \lambda \in \mathbb{R}\}$. Then the **relative volume** of S is the volume computed relative

to the sublattice (span S) $\cap \mathbb{Z}^d$.

Theorem 2.12 looks only at parallelepipeds whose dimension matches the dimension of the space. Thus, we need a generalization of this theorem [2].

Lemma 2.14. *Suppose $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n \in \mathbb{Z}^d$ are linearly independent, and let*

$$\Pi := \{\lambda_1 \boldsymbol{w}_1 + \lambda_2 \boldsymbol{w}_2 + \cdots + \lambda_n \boldsymbol{w}_n : 0 \leq \lambda_j < 1\},$$

and let V be the greatest common divisor of all $n \times n$ minors of the matrix formed by the column vectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$. Then the relative volume of Π equals V. Furthermore,

$$\#(\Pi \cap \mathbb{Z}^d) = V,$$

and for every positive integer t,

$$\#(t\Pi \cap \mathbb{Z}^d) = Vt^d.$$

We thus get the following theorem about the Ehrhart polynomials of zonotopes:

Theorem 2.15 ([7]). *Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n \in \mathbb{Z}^d$ and $\mathcal{Z}$ be the zonotope generated by $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. Then*

$$L_{\mathcal{Z}}(t) = \sum_S m(S) t^{|S|},$$

where S ranges over all linearly independent subsets of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, and $m(S)$ is the gcd of all minors of size $|S|$ of the matrix whose columns are the elements of S.

The next logical step, given the outline of the paper so far, is to ask, "Well, what about rational zonotopes?" From Ehrhart's theorem on rational polytopes, Theorem 2.7, we know that the counting function is a quasipolynomial in t of degree d whose period divides the least common multiple of the denominators of the coordinates of the vertices of the zonotope, but can we say anything else? Can we use the decomposition into half-open rational parallelepipeds to give us a nice theorem like we had for integral zonotopes? The answer is that we do not yet know. There is, as of yet, no such theorem, perhaps due to the complexity of determining the Ehrhart quasipolynomial of even just a parallelepiped. We can still look at a couple examples of rational zonotopes—in particular, rational cubes—to see what we can come up with.

2.6 Examples of Rational Cubes

2.6.1 Rational Generators

The lovely unit square we examined at the start of the paper will no longer work as an interesting example, given that the coordinates of its vertices are all integral; consider, instead, $\square_{\frac{1}{5}} = \left[0, \frac{1}{5}\right]^2$. Figure 2.12 shows this square and its 1st through

13th dilates. The dilates are shaded from light to dark to help us see the pattern: the dilates that are congruent to 0 mod 5 are the lightest, and those that are congruent to 4 mod 5 are the darkest. This pattern suggests that the period of our quasipolynomial is 5.

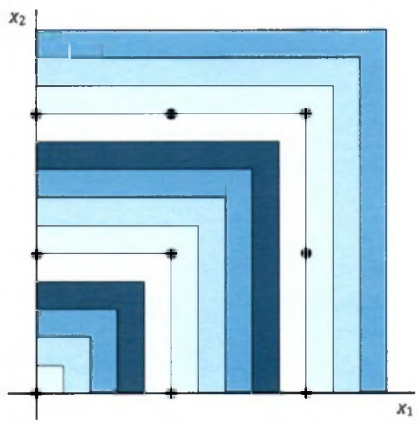

Figure 2.12: Some dilates of $\left[0, \frac{1}{5}\right]^2$.

Notice that every 5th dilate of $\square_{\frac{1}{5}}$ is a dilate of the unit square; we already know the Ehrhart polynomial for that, so we can modify it slightly so that it works for the 0 mod 5 dilates of $\square_{\frac{1}{5}}$. Instead of the 1st, 2nd, 3rd, etc. dilates, we get the unit square dilates at the 5th, 10th, 15th, etc. dilates of $\square_{\frac{1}{5}}$, so if we divide the dilate input by 5, we should have what we are looking for. That is, for $t \equiv 0 \bmod 5$,

$$L_{\square_{\frac{1}{5}}}(t) = \left(\frac{1}{5}t + 1\right)^2.$$

One constituent down, four to go.

Looking at Figure 2.12, we see that it is only the 0 mod 5 dilates that capture new integer points; all other dilates keep the same number of points as whichever 0 mod 5 dilate came directly before. This indicates that we can modify the polynomial we got for the 0 mod 5 case slightly to give us our desired polynomials. This time, though, we are not working with multiples of 5, so we cannot just divide by 5. Consider the dilates for $t \equiv i \bmod 5$; subtracting i from t yields a multiple of 5.

First, let us assume $t < 5$. There is exactly 1 integer point in this dilate, and we can capture the 1 from this point by subtracting i from t, which gives 0; dividing by 5; adding 1; and squaring the resulting 1.

What about $5 < t < 10$? We should get 4 points, and we do so by subtracting i from t, which yields 5; dividing by 5; adding 1; and squaring the resulting 2, which gives us the desired 4. This pattern continues to hold for $t > 10$.

Given the connection to the unit square, we deduce that the Ehrhart quasipolynomial for $\square_{\frac{1}{5}}$ is

$$L_{\square_{\frac{1}{5}}}(t) = \begin{cases} \left(\frac{1}{5}(t-0)+1\right)^2 & \text{if } t \equiv 0 \bmod 5, \\ \left(\frac{1}{5}(t-1)+1\right)^2 & \text{if } t \equiv 1 \bmod 5, \\ \left(\frac{1}{5}(t-2)+1\right)^2 & \text{if } t \equiv 2 \bmod 5, \\ \left(\frac{1}{5}(t-3)+1\right)^2 & \text{if } t \equiv 3 \bmod 5, \\ \left(\frac{1}{5}(t-4)+1\right)^2 & \text{if } t \equiv 4 \bmod 5. \end{cases}$$

Given the nature of cubes, we generalize this quasipolynomial for $\left[0, \frac{1}{n}\right]^d$ by replacing the 2s with ds and the 5s with ns:

$$L_{\left[0,\frac{1}{n}\right]^d}(t) = \begin{cases} \left(\frac{1}{n}(t-0)+1\right)^d & \text{if } t \equiv 0 \bmod n, \\ \left(\frac{1}{n}(t-1)+1\right)^d & \text{if } t \equiv 1 \bmod n, \\ \quad\vdots & \\ \left(\frac{1}{n}(t-i)+1\right)^d & \text{if } t \equiv i \bmod n, \\ \quad\vdots & \\ \left(\frac{1}{n}(t-(n-1))+1\right)^d & \text{if } t \equiv n-1 \bmod n. \end{cases}$$

The Ehrhart series for $\left[0, \frac{1}{n}\right]^d$ will have n different parts as well, but these parts can be put together. We will use the $t \equiv i \bmod n$ case to compute the Ehrhart series.

$$\begin{aligned}
\sum_{t\equiv i \bmod n} L_{[0,\frac{1}{n}]^d}(t)z^t &= \sum_{t\equiv i \bmod n} \left(\frac{1}{n}(t-i)+1\right)^d z^t \\
&= \sum_{r\geq 0} \left(\frac{1}{n}(nr+i-i)+1\right)^d z^{nr+i} \\
&= \sum_{r\geq 0} (r+1)^d z^{nr} z^i \\
&= z^i \sum_{r\geq 0} (r+1)^d (z^n)^r \\
&= \frac{z^i}{z^n} \sum_{r\geq 1} r^d (z^n)^r \\
&= \frac{z^i \sum_{k=1}^{d} A(d,k)(z^n)^{k-1}}{(1-z^n)^{d+1}}.
\end{aligned}$$

Since our choice of i is arbitrary, putting all choices together gives us

$$\begin{aligned}
\mathrm{Ehr}_{[0,\frac{1}{n}]^d}(z) &= \frac{\sum_{i=0}^{n-1} z^i \left(\sum_{k=1}^{d} A(d,k)(z^n)^{k-1}\right)}{(1-z^n)^{d+1}} \\
&= \frac{\left(\frac{1-z^n}{1-z}\right)\left(\sum_{k=1}^{d} A(d,k)(z^n)^{k-1}\right)}{(1-z^n)^{d+1}} \\
&= \frac{\sum_{k=1}^{d} A(d,k)(z^n)^{k-1}}{(1-z)(1-z^n)^d}.
\end{aligned}$$

An interesting observation is that this Ehrhart series is remarkably similar to that

of the unit cube:

$$\mathrm{Ehr}_{[0,1]^d}(z) = \frac{\sum_{k=1}^{d} A(d,k) z^{k-1}}{(1-z)^{d+1}}.$$

Do all rational cubes have this similar structure? What about cubes of the form $\left[0, \frac{k}{n}\right]$? As an example, consider $\left[0, \frac{3}{5}\right]^2$, pictured in Figure 2.13. As with Figure 2.12, the picture of some dilates of $\left[0, \frac{1}{5}\right]^2$, the dilates are shaded according to their value mod5.

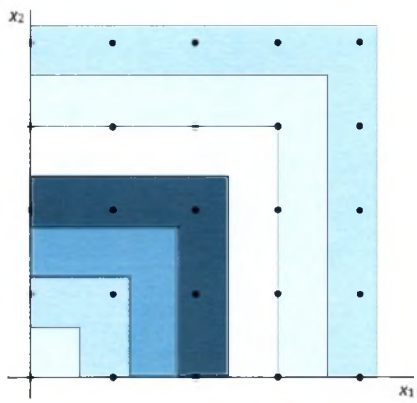

Figure 2.13: Some dilates of $\left[0, \frac{3}{5}\right]^2$.

This square is a little trickier than the previous example, because now new points show up in dilates other than the 0 mod 5 dilates. The 0 mod 5 dilates are still straightforward enough: for $t \equiv 0 \bmod 5$,

$$L_{\left[0,\frac{3}{5}\right]^2}(t) = \left(\frac{1}{5}(3t) + 1\right)^2.$$

The question, then, is, "How many points are on the line segment $\left[0, \frac{3t}{5}\right]$?" The

answer, for $t \equiv i \bmod 5$, is $\frac{1}{5}(3t - (3i \bmod 5)) + 1$. We subtract $3i \bmod 5$ from $3t$ to achieve the same effect that subtracting i from 5 had in the previous example: to bring us down to the last integer point. Dividing by 5 takes away all the noninteger points, and adding 1 ensures that the origin is counted. Squaring this result for each value of i gives us the Ehrhart quasipolynomial:

$$L_{[0,\frac{3}{5}]^2}(t) = \begin{cases} \left(\frac{1}{5}(3t - (0 \times 3 \bmod 5)) + 1\right)^2 & \text{if } t \equiv 0 \bmod 5, \\ \left(\frac{1}{5}(3t - (1 \times 3 \bmod 5)) + 1\right)^2 & \text{if } t \equiv 1 \bmod 5, \\ \left(\frac{1}{5}(3t - (2 \times 3 \bmod 5)) + 1\right)^2 & \text{if } t \equiv 2 \bmod 5, \\ \left(\frac{1}{5}(3t - (3 \times 3 \bmod 5)) + 1\right)^2 & \text{if } t \equiv 3 \bmod 5, \\ \left(\frac{1}{5}(3t - (4 \times 3 \bmod 5)) + 1\right)^2 & \text{if } t \equiv 4 \bmod 5. \end{cases}$$

Notice that the number of integer points on $\left[0, \frac{k}{n}\right]$ is $\frac{1}{n}(kt - (ki \bmod n)) + 1$, so raising this expression to the dth power would give us the number of integer points in the tth dilate of $\left[0, \frac{k}{n}\right]^d$ for $t \equiv i \bmod n$. Thus we have another general Ehrhart quasipolynomial:

$$L_{[0,\frac{k}{n}]^d}(t) = \begin{cases} \left(\frac{1}{n}(kt - (0 \times k \bmod n)) + 1\right)^d & \text{if } t \equiv 0 \bmod n, \\ \left(\frac{1}{n}(kt - (1 \times k \bmod n)) + 1\right)^d & \text{if } t \equiv 1 \bmod n, \\ \vdots & \\ \left(\frac{1}{n}(kt - (i \times k \bmod n)) + 1\right)^d & \text{if } t \equiv i \bmod n, \\ \vdots & \\ \left(\frac{1}{n}(ki - ((n-1) \times k \bmod n)) + 1\right)^d & \text{if } t \equiv n-1 \bmod n. \end{cases}$$

As before, we can compute the Ehrhart series for one value of i and then add them all together.

$$\begin{aligned} \sum_{t \equiv i \bmod n} L_{[0,\frac{k}{n}]^d}(t) z^t &= \sum_{t \equiv i \bmod n} \left(\frac{1}{n}(kt - (i \times k \bmod n)) + 1\right)^d z^t \\ &= \sum_{r \geq 0} \left(\frac{1}{n}(k(nr+i) - (i \times k \bmod n)) + 1\right)^d z^{nr+i} \\ &= \sum_{r \geq 0} \left(\frac{1}{n}(knr + ki - (i \times k \bmod n)) + 1\right)^d z^{nr+i}. \end{aligned}$$

Note that $\frac{ki-(i\times k \bmod n)}{n}$ is $\lfloor \frac{ki}{n} \rfloor$, a constant. Let $\alpha = \lfloor \frac{ki}{n} \rfloor + 1$. Then the previous expression can be further simplified.

$$\begin{aligned}
&\sum_{r\geq 0}\left(\frac{1}{n}(knr+ki-(i\times k \bmod n))+1\right)^d z^{nr+i}\\
&\quad=\sum_{r\geq 0}\left(kr+\frac{ki-(i\times k \bmod n)}{n}+1\right)^d z^{nr+i}\\
&\quad=\sum_{r\geq 0}\left(kr+\left\lfloor\frac{ki}{n}\right\rfloor+1\right)^d z^{nr+i}\\
&\quad=\sum_{r\geq 0}(kr+\alpha)^d z^{nr+i}. \qquad (2.1)
\end{aligned}$$

However, we can no longer use our little reindexing trick from before that gave us $\sum_{r\geq 0} r^d z^r$, but if we use the binomial expansion of $(kr+\alpha)^d$, we might be able to get r by itself.

$$\begin{aligned}
(kr+\alpha)^d &= \sum_{m=0}^{d}\binom{d}{m}(kr)^m\alpha^{d-m}\\
&= \sum_{m=0}^{d}\left(\binom{d}{m}k^m\alpha^{d-m}\right)r^m. \qquad (2.2)
\end{aligned}$$

We then substitute (2.2) into (2.1):

$$\begin{aligned}
\sum_{r\geq 0}((kr+\alpha)^d z^{nr+i} &= \sum_{r\geq 0}\left(\sum_{m=0}^{d}\left(\binom{d}{m}k^m\alpha^{d-m}\right)r^m\right)z^{nr+i} \\
&= \sum_{m=0}^{d}\left(\binom{d}{m}k^m\alpha^{d-m}\sum_{r\geq 0}r^m z^{nr+i}\right) \\
&= z^i\sum_{m=0}^{d}\left(\binom{d}{m}k^m\alpha^{d-m}\sum_{r\geq 0}r^m(z^n)^r\right) \\
&= z^i\sum_{m=0}^{d}\left(\binom{d}{m}k^m\alpha^{d-m}\frac{\sum_{s=0}^{m}A(m,s)(z^n)^s}{(1-z^n)^{m+1}}\right).
\end{aligned}$$

Adding the parts of the series corresponding to each i together gives us our Ehrhart series:

$$\mathrm{Ehr}_{\left[0,\frac{k}{n}\right]^d}(z) = \sum_{i=0}^{n-1}\left(z^i\sum_{m=0}^{d}\left(\binom{d}{m}k^m\alpha^{d-m}\frac{\sum_{s=0}^{m}A(m,s)(z^n)^s}{(1-z^n)^{m+1}}\right)\right).$$

While the Eulerian polynomials show up again, this Ehrhart series is much more complicated than that of $\left[0,\frac{1}{n}\right]^d$.

2.6.2 A Shifted Cube

Using rational vectors to generate a zonotope is not the only way to get a rational zonotope; another method we can apply is translating the zonotope by a rational vector. For example, consider our first example: the unit square, $\square_2$. Now, consider $\square_2 - \frac{1}{2}$, shown in Figure 2.14. The vertices of our shifted cube all have denominator

2, so we expect, from Ehrhart's Theorem for rational polytopes (Theorem 2.7), that our counting function is a quasipolynomial with period 1 or 2.

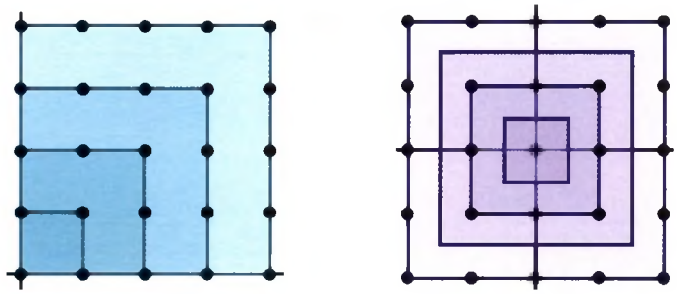

Figure 2.14: Dilates of $\square_2$ (left) and of $\square_2 - \frac{1}{2}$ (right).

The even dilates are integral and thus have the same Ehrhart polynomial as $\square_2$,

$$L_{\square_2}(t) = (t+1)^2.$$

Do the odd dilates follow the same pattern? The first dilate has 1 point, not 4 points, so our answer is no, they do not; however, the pattern is similar. The first dilate, as mentioned, has 1 point, and the third dilate has 9 points. More generally, when t is odd, the tth dilate contains t^2 integer points. Thus our Ehrhart quasipolynomial is

$$L_{\square_2 - \frac{1}{2}}(t) = \begin{cases} (t+1)^2 & \text{if } t \text{ even,} \\ t^2 & \text{if } t \text{ odd.} \end{cases}$$

2.7 Coxeter Permutahedra

We now introduce another flavor of zonotope: a permutahedron. We begin by defining the **classical root systems** [1], considering only the positive roots.

$$\begin{aligned}
\boldsymbol{A_{d-1}} &= \{e_i - e_j : 1 \leq i < j \leq d\} \\
\boldsymbol{B_d} &= \{e_i - e_j,\ e_i + e_j : 1 \leq i < j \leq d\} \cup \{e_i : 1 \leq i \leq d\} \\
\boldsymbol{C_d} &= \{e_i - e_j,\ e_i + e_j : 1 \leq i < j \leq d\} \cup \{2e_i : 1 \leq i \leq d\} \\
\boldsymbol{D_d} &= \{e_i - e_j,\ e_i + e_j : 1 \leq i < j \leq d\}.
\end{aligned}$$

A natural thing for us to do is to consider these vectors as generators of a zonotope. These are called Coxeter permutahedra. For example, $\mathcal{Z}(B_d)$ is the zonotope with generators from B_d, and we call $\mathcal{Z}(B_d)$ the type-B permutahedron. Figure 2.15 shows the $d = 2$ case for each type.

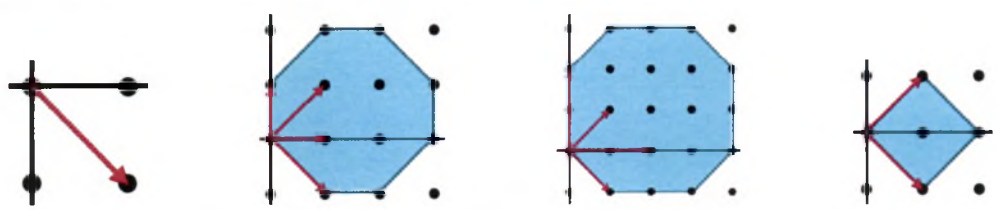

Figure 2.15: $\mathcal{Z}(A_{2-1})$, $\mathcal{Z}(B_2)$, $\mathcal{Z}(C_2)$, and $\mathcal{Z}(D_2)$.

The Ehrhart polynomials of each of these lattice permutahedra are computed in [1]. Proposition 2.16 indicates why these are called "permutahedra."

Proposition 2.16. $\mathcal{Z}(A_{d-1}) = \text{conv}(\textit{permutations of } \{0, 1, \ldots, d-1\})$.

2.8 Signed Graphs

Before continuing on our journey of Ehrhart theory and zonotopes, we need to take a quick detour through the land of graph theory. In particular, signed graphs will prove to be a useful tool in Chapter 3. Graphs provide a way to keep track of connections between objects. See [9] for more information on graphs.

A **graph** G is a pair $G = (V, E)$ consisting of a set V of **nodes** (or vertices) and a set E of 1- or 2-element subsets of V, which we call **edges**. An edge of the form ij, a 2-element subset of V, is a **link**, and an edge of the form i, a 1-element subset of V, is a **half edge**. Our graphs will have neither loops nor multiple edges. The **degree** of a node is the number of edges going into that node. A degree-one node is called a **leaf**.

A **subgraph** of a graph $G = (V, E)$ is a graph $G' = (V', E')$ for which $V' \subseteq V$ and $E' \subseteq E$ such that E' consists of 1- and 2-element subsets of V'. A **path** is a non-empty graph $P = (V, E)$ where

$$V = \{n_1, n_2, \ldots, n_k\}$$

and

$$E = \{n_1 n_2, n_2 n_3, \ldots, n_{k-1} n_k\},$$

where $n_1, \ldots, n_k$ are distinct. A **cycle** is a non-empty graph $C = (V, E)$ where

$$V = \{n_1, n_2, \ldots, n_k\}$$

and

$$E = \{n_1n_2, n_2n_3, \ldots, n_{k-1}n_k, n_kn_1\},$$

where $n_1, \ldots, n_k$ are distinct. We will sometimes refer to "a cycle on $[k]$," by which

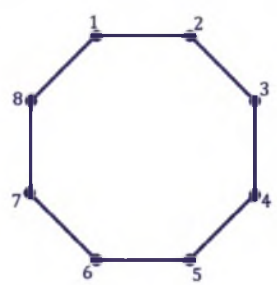

Figure 2.16: A cycle on [8].

we mean a cycle with k nodes labeled $1, 2, \ldots, k$, such as in Figure 2.16, with $k = 8$. A graph is **connected** if there exists a path between any two of its nodes. A **connected component** of a graph is a maximal connected subgraph. A **tree** is a connected graph with no cycles and no half edges. A graph is a **forest** if each of its connected components is a tree.

We need something more general than graphs, and this leads us to consider signed graphs. A **signed graph** S is a pair $S = (G, \sigma)$ consisting of a graph G and a sign function $\sigma : E \to \{-1, +1\}$ that labels each link with a $-$ sign or a $+$ sign. Thus S has three types of edges: positive edges, negative edges, and half edges.

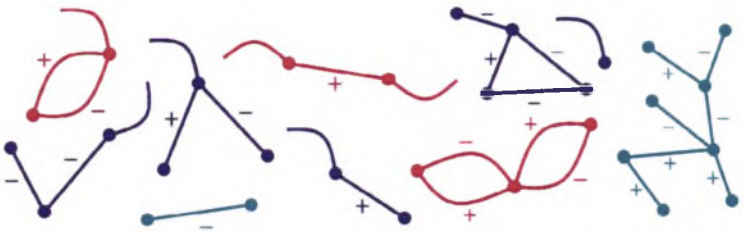

Figure 2.17: Some examples of signed graphs.

A cycle is **balanced** if the product of the signs on its edges is positive and **unbalanced** if the product of the signs on its edges is negative (that is, if the cycle is not balanced). Figure 2.18 gives examples of graphs containing both balanced and unbalanced cycles.

Figure 2.18: Graphs with balanced cycles (left) and unbalanced cycles (right).

We define one more term that we will use later: a tree convex hull.

Figure 2.19: The tree convex hull of the even-degree nodes.

Definition 2.2. The **tree convex hull** of nodes $n_1, n_2, \ldots, n_k$ is the union of all paths joining these nodes.

Figure 2.8 gives an example of a tree convex hull.

Chapter 3

Rational Coxeter Permutahedra

As mentioned in Section 2.5, it is sometimes useful to translate a zonotope so that its center of mass is at the origin. The vertices of a translated lattice zonotope may stay on the lattice, or they may be shifted off. For instance, we can see in Figure 3.1 that the 2-dimensional type-C permutahedron stays on the lattice whereas the 2-dimensional type-B permutahedron is shifted off. Let $\widetilde{\mathcal{Z}}$ denote the translated zonotope with center of mass at the origin.

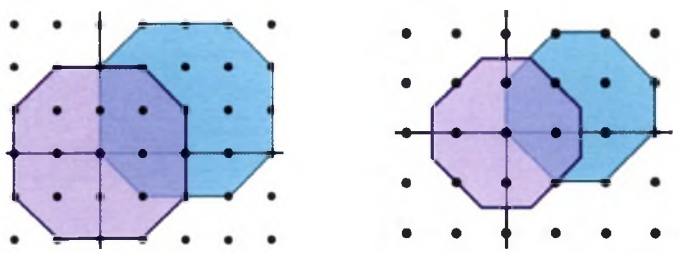

Figure 3.1: Upon translation, the vertices of $\widetilde{\mathcal{Z}}(C_2)$ (left) stay on the lattice, but the vertices of $\widetilde{\mathcal{Z}}(B_2)$ (right) are shifted off.

3.1 Type B

We begin our adventure into rational permutahedra with $\mathcal{Z}(B_d)$. As seen in Figure 3.1, $\widetilde{\mathcal{Z}}(B_2)$ is half-integral; specifically, its vertices are all rational with denominator 2. Thus (Figure 3.2) the second and fourth (and, more generally, the even) dilates of $\widetilde{\mathcal{Z}}(B_2)$ are integral.

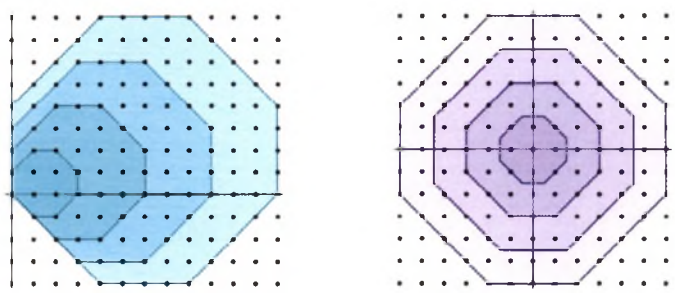

Figure 3.2: Some dilates of $\mathcal{Z}(B_2)$ and $\widetilde{\mathcal{Z}}(B_2)$.

Ehrhart's Theorem 2.6 tells us that the Ehrhart polynomial of $\mathcal{Z}(B_2)$ is quadratic. Furthermore, Theorem 2.9 tells us that the leading coefficient is $\mathrm{vol}(\mathcal{Z}(B_2)) = 7$, and the constant term is 1. Using the fact that the discrete volume of the first dilate is 12, we can determine the Ehrhart polynomial:

$$L_{\mathcal{Z}(B_2)}(t) = 7t^2 + 4t + 1. \tag{3.1}$$

Ehrhart's Theorem 2.7 for rational polytopes tells us that the Ehrhart quasipolynomial for $\widetilde{\mathcal{Z}}(B_2)$ has degree 2 as well. The period of this quasipolynomial is 2, which we can see from the even dilates being integral. Thus we need only focus on the odd

dilates. The leading coefficient is still 7, the first dilate has 9 points, and the third dilate has 69 points. Putting this all together, we can determine the quadratic for the odd dilates and thus the Ehrhart quasipolynomial for $\widetilde{\mathcal{Z}}(B_2)$:

$$L_{\widetilde{\mathcal{Z}}(B_2)}(t) = \begin{cases} 7t^2 + 4t + 1 & \text{if } t \text{ even,} \\ 7t^2 + 2t & \text{if } t \text{ odd.} \end{cases} \tag{3.2}$$

We could repeat this process for $\widetilde{\mathcal{Z}}(B_3)$ and higher, but this process is not really feasible for general d. Our goal is to compute the Ehrhart quasipolynomial of $\widetilde{\mathcal{Z}}(B_d)$ for any d.

Since a zonotope can be tiled by half-open parallelepipeds (2.11), it seems a logical next step to look at linearly independent subsets of the generating vectors. For B_2, this is straight-forward: any combination of two or fewer vectors in B_2, that is, $\binom{1}{0}$, $\binom{0}{1}$, $\binom{1}{1}$, and $\binom{1}{-1}$, is linearly independent.

For B_d with $d > 2$, it is not the case that any combination of d vectors is linearly independent; for instance, A_2 is a subset of B_3 that has 3 vectors and spans a 2-dimensional space. To help us determine which sets are linearly independent, we turn to signed graphs, using the following construction from [8].

Definition 3.1. Let S be a subset of B_d. We construct the corresponding signed graph $\boldsymbol{G_S}$ on $[d]$ by including:

- the positive edge ij for each vector $e_i - e_j \in S$;

- the negative edge ij for each vector $e_i + e_j \in S$;
- the half edge i for each vector $e_i \in S$.

Lemma 3.1 ([8]). *The subsets of B_d are in bijection with signed graphs on $[d]$.*

Proof. This follows from Definition 3.1 □

Figure 3.3 shows all six subsets of two vectors of B_2.

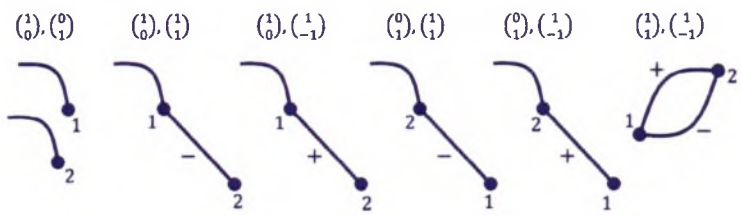

Figure 3.3: Sets of 2 linearly independent vectors from B_2 and their corresponding signed graphs.

3.1.1 Linear Independence

One observation we can make is that, since B_d lives in $\mathbb{R}^d$, any set of more than d vectors cannot be linearly independent. The number of vectors in $S \subset B_d$ is precisely the number of edges of the graph G_S; in order for S to be linearly independent, then, G_S can have at most d edges.

Similarly, a graph with n nodes and more than n edges corresponds to a set of linearly dependent vectors. We define the following types of connected components:

Definition 3.2. Let G_S be a signed graph.

- A **cycle component (CC)** of G_S is a connected component of G_S that contains a single cycle, which is unbalanced, and no half edges; $\boldsymbol{cc(G)}$ is the number of cycle components of G.
- A **half edge component (HC)** of G_S is a connected component of G_S that contains a single half edge and no cycles; $\boldsymbol{hc(G)}$ is the number of half edge components of G.
- A **tree component (TC)** of G_S is a connected component of G_S that is a tree; $\boldsymbol{tc(G)}$ is the number of tree components of G.

Theorem 3.2 ([9]). *Subsets of B_d are linearly independent if and only if their corresponding graph contains only CC, HC, and TC; equivalently, every component of the graph has at least as many nodes as edges, and all cycles are unbalanced.*

Proof. Let $S \subset B_d$ and G_S be its corresponding signed graph. We have already seen that the connected components of G_S must have at least as many nodes as edges in order for S to be linearly independent. Possible types of connected components, then, are tree components, half edge components, and components containing a single cycle (more than one cycle would give us more edges than nodes, as would a cycle with a half edge or a connected component with multiple half edges).

Claim 1: A tree component corresponds to a linearly independent set of vectors. We shall proceed by induction.

Consider a tree on 2 nodes. Regardless of the sign on the edge, this tree has one edge and thus corresponds to a single vector with 2 nonzero entries, which is linearly independent.

Suppose, now, that every tree on n nodes corresponds to a linearly independent set of vectors and consider a tree on $n+1$ nodes. There exists a leaf; say that this is the $(n+1)$st node. Thus there is only one vector that has a nonzero entry in the $(n+1)$st component. Consider the tree formed by removing this node and the edge connected to it. We are left with a tree on n nodes, which corresponds to a linearly independent set of vectors, all of which have an entry of 0 in the $(n+1)$st component. If we add the removed vector back to this set, our new set of vectors will still be linearly independent. Thus a tree on $n+1$ nodes corresponds to a linearly independent set of vectors.

Thus a tree component corresponds to a set of linearly independent vectors.

Claim 2: A half edge component corresponds to a linearly independent set of vectors.

As with TC, we shall proceed by induction.

Consider the graph that is a single half edge. This half edge corresponds to a single vector with 1 nonzero entry, which is linearly independent.

Suppose, now, that every half edge component on n nodes corresponds to a linearly independent set of vectors and consider an HC on $n+1$ nodes. As with the TC, there exists a leaf. Applying the same steps as with the TC shows that an HC

on $n+1$ nodes corresponds to a linearly independent set of vectors.

Thus a half edge component corresponds to a linearly independent set of vectors.

Claim 3: A connected component containing a balanced cycle corresponds to a linearly dependent set of vectors.

Consider a balanced cycle on $[n]$. We can assume, without loss of generality, that our edges are $j(j+1)$ with node $n+1$ being the same as node 1. Let σ_j be the sign on edge $j(j+1)$; then σ_j is either 1 or -1. See Figure 3.4 for an example of such a cycle.

Since our cycle is balanced, $\prod_{j=1}^{n} \sigma_j = +1$.

Our vectors are linearly dependent if we can find some nonzero linear combination of our vectors that is equal to zero; that is,

$$\lambda_1(e_1 - \sigma_1 e_2) + \lambda_2(e_2 - \sigma_2 e_3) + \cdots + \lambda_n(e_n - \sigma_n e_1) = 0$$

with some $\lambda_j \neq 0$.

Rearranging the left hand side to put the unit vectors together yields

$$\begin{aligned}
&\lambda_1(e_1 - \sigma_1 e_2) + \lambda_2(e_2 - \sigma_2 e_3) + \cdots + \lambda_n(e_n - \sigma_n e_1) \\
&= (\lambda_1 - \sigma_n \lambda_n)e_1 + (\lambda_2 - \sigma_1 \lambda_1)e_2 + \cdots + (\lambda_n - \sigma_{n-1}\lambda_{n-1})e_n \\
&= \sum_{j=1}^{n} (\lambda_{j+1} - \sigma_j \lambda_j) e_{j+1},
\end{aligned}$$

with $\lambda_{n+1} = \lambda_1$ and $e_{n+1} = e_1$.

In order for this sum to equal 0, we need $\lambda_{j+1} - \sigma_j \lambda_j = 0$, that is, $\lambda_{j+1} = \sigma_j \lambda_j$. We may assume $\lambda_1 = 1$. Then

$$\begin{aligned} \lambda_2 &= \sigma_1 \\ \Rightarrow \lambda_3 &= \sigma_2 \sigma_1 \\ &\vdots \\ \Rightarrow \lambda_j &= \sigma_{j-1} \cdots \sigma_1. \end{aligned}$$

In particular, $\lambda_n = \sigma_{n-1} \cdots \sigma_1$. Thus

$$\begin{aligned} \lambda_1 - \sigma_n \lambda_n &= 1 - \sigma_n(\sigma_{n-1} \cdots \sigma_1) \\ &= 0. \end{aligned}$$

Thus our set of vectors is linearly dependent.

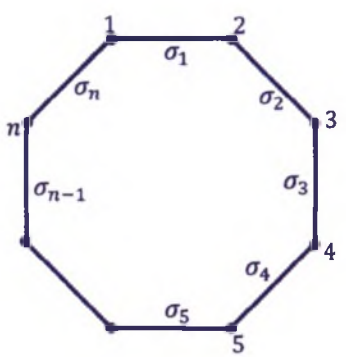

Figure 3.4: A signed cycle on $[n]$ with sign σ_j on edge $j(j+1)$.

Claim 4: A cycle component corresponds to a linearly independent set of vectors.

Consider an unbalanced cycle on $[n]$. We use the same setup as in the previous claim, using Figure 3.4 for reference. Since our cycle is unbalanced, $\prod_{j=1}^{n} \sigma_j = -1$. We need to show that, for any linear dependence,

$$\begin{aligned} 0 &= \lambda_1(e_1 - \sigma_1 e_2) + \lambda_2(e_2 - \sigma_2 e_3) + \cdots + \lambda_n(e_n - \sigma_n e_1) \\ &= \sum_{j=1}^{n} (\lambda_{j+1} - \sigma_j \lambda_j) e_{j+1} \end{aligned}$$

we have $\lambda_j = 0$ for all j, with $\lambda_{n+1} = \lambda_1$. As before, in order for this sum to equal 0, we need $\lambda_{j+1} = \sigma_j \lambda_j$.

We now claim $\lambda_1 = 0$.

$$\begin{aligned} \lambda_2 &= \sigma_1 \lambda_1 \\ \Rightarrow \lambda_3 &= \sigma_2 \sigma_1 \lambda_1 \\ &\vdots \\ \Rightarrow \lambda_j &= \sigma_{j-1} \ldots \sigma_1 \lambda_1. \end{aligned}$$

In particular, $\lambda_n = \sigma_{n-1} \dots \sigma_1 \lambda_1$. Thus

$$\begin{aligned} 0 &= \lambda_1 - \sigma_n \lambda_n \\ &= \lambda_1 - \sigma_n(\sigma_{n-1} \dots \sigma_1 \lambda_1) \\ &= \lambda_1 - \sigma_n \dots \sigma_1 \lambda_1 \\ &= \lambda_1 + \lambda_1. \end{aligned}$$

Thus

$$\begin{aligned} \lambda_1 &= 0 \\ \Rightarrow \lambda_2 &= 0 \\ &\vdots \\ \Rightarrow \lambda_n &= 0. \end{aligned}$$

Thus our set of vectors is linearly independent. □

Definition 3.3. We call the signed graph G_S **independent** if S is linearly independent.

3.1.2 Volume

Lemma 3.3 ([1]). *Let S be a linearly independent subset of B_d and G_S be its corresponding graph. The relative volume of $\mathcal{Z}(S)$ is $2^{cc(G_S)}$.*

Proof. In order to determine the relative volume of $\mathcal{Z}(S)$, we can look at the parts of S that correspond to connected components of G_S, find the relative volumes of those subsets, and multiply the volumes together. One thing to note is that each of these connected components corresponds to a parallelepiped, and the number of vertices of an n-dimensional parallelepiped is 2^n. Since all of our vectors are integral, each vertex will also be integral. In particular, say $P = \{\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n : \lambda_j \in [0,1]\}$; these 2^n vertices occur when $\lambda_j \in \{0,1\}$ for each $j = 1, \ldots, n$. The question, then, is whether or not we can get an integral point when not all λ_j are precisely 0 or 1.

Each $\mathbf{v}_j$ corresponds to an edge in G_S by Definition 3.1, and each λ_j becomes a label on the edge. In order for the ith index of $\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_n\mathbf{v}_n$ to be integral, we need the sum of the labels of the edges of the form ij, ji, and i to be integral. Let $L(i)$ be this sum.

TC: We claim that, for a subset $S \subset B_d$ whose graph G_S is a tree component, the only integer points of $\mathcal{Z}(S)$ are its vertices; that is, every edge of G_S receives an integer label. We shall proceed by induction.

Suppose G_S is a tree on 2 nodes. This tree has one edge, which needs an integral label — either 0 or 1 — in order for $L(1)$ and $L(2)$ to be integral; thus the only integer points in $\mathcal{Z}(S)$ are its vertices.

Suppose, now, that the edges of every tree on n nodes need integral labels in order for $L(i)$ to be integral for every node i. Suppose, then, that G_S is a tree on

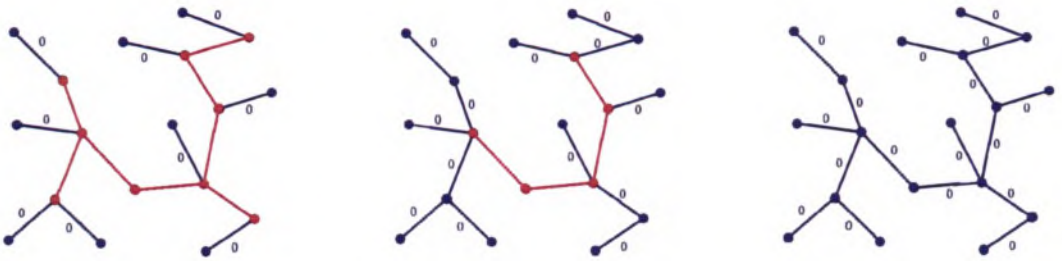

Figure 3.5: Labeling the tree so that each node is integral.

$n+1$ nodes. G_S has at least one leaf; call it x, and say that the edge connected to node x is xy. Then xy needs an integral label, so we label it 0. Now, this edge label contributes an integer value to $L(y)$, thus we can remove edge xy and node x without changing the rest of the edge labels. We now have a tree on n nodes, which needs integral labels. Thus the only integer points in $\mathcal{Z}(S)$ are its vertices.

Thus the relative volume of $\mathcal{Z}(S)$, where G_S is a tree component, is 1.

HC: We claim that, for a subset $S \subset B_d$ whose graph G_S is a half edge component, the only integer points of $\mathcal{Z}(S)$ are its vertices; that is, every edge of G_S receives an integer label. As with TC, we shall proceed by induction.

Consider the graph G_S that has a single half edge. This half edge must have an integral label — again, either 0 or 1 — in order for $L(1)$ to be integral; thus the only integer points in $\mathcal{Z}(S)$ are its vertices.

Suppose, now, that the edges of every HC on n nodes needs integral labels in order for $L(i)$ to be integral for every node i. Suppose, then, that G_S is a half edge component on $n+1$ nodes. As with the TC, there exists a leaf. Applying the same

steps as with the TC shows that every edge of G_S needs integral labels; thus the only integer points in $\mathcal{Z}(S)$ are its vertices.

Thus the relative volume of $\mathcal{Z}(S)$, where G_S is a half edge component, is 1.

CC: We claim that, for a subset $S \subset B_d$ whose graph G_S is a cycle component, the only integer points of $\mathcal{Z}(S)$ occur when G_S is labeled such that the non-cycle edges receive integer labels and the cycle edges receive labels from $\{0, 1, \frac{1}{2}\}$.

Suppose G is a cycle component. There are two options: either G is just a cycle, or G is a cycle with more edges coming out of it, and the set of edges that stem from node i form a tree for each node on the cycle, as demonstrated in Figure 3.6. Applying the same argument as in TC shows that these edges that are not a part of the cycle need integer labels.

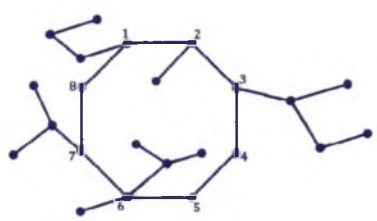

Figure 3.6: A graph that contains a cycle and edges that are not a part of the cycle.

Thus it suffices to consider an unbalanced cycle on n nodes. Let each $\lambda_j = \frac{1}{2}$, as

shown in Figure 3.7; then for each node i on the cycle, our options are

$$L(i) = \frac{1}{2} + \frac{1}{2} = 1$$
$$L(i) = \frac{1}{2} - \frac{1}{2} = 0$$
$$L(i) = -\frac{1}{2} - \frac{1}{2} = -1.$$

In any case, $L(i)$ is integral. Thus we have at least one integer point in $\mathcal{Z}(S)$ that is not a vertex.

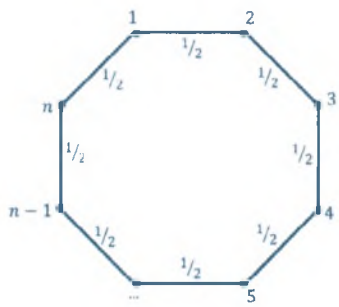

Figure 3.7: A cycle with all $\lambda_j = \frac{1}{2}$.

Claim: The only non-integer labeling of the cycle edges that yields an integer point stems from all edges being labeled $\frac{1}{2}$.

We shall make use of the notation we used in the proof of Lemma 3.2; that is, the edge $j(j+1)$ has sign σ_j. Thus we need

$$\lambda_1(e_1 - \sigma_1 e_2) + \lambda_2(e_2 - \sigma_2 e_3) + \cdots + \lambda_{n-1}(e_{n-1} - \sigma_{n-1} e_n) + \lambda_n(e_1 - \sigma_n e_n) \in \mathbb{Z}^n.$$

Rearranging the left hand side gives

$$(\lambda_1+\lambda_n)e_1+(\lambda_2-\sigma_1\lambda_1)e_2+\cdots+(\lambda_{n-1}-\sigma_{n-2}\lambda_{n-2})e_{n-1}+(-\sigma_{n-1}\lambda_{n-1}-\sigma_n\lambda_n)e_n$$
$$=\sum_{j=2}^{n-1}(\lambda_j-\sigma_{j-1}\lambda_{j-1})e_j+(\lambda_1+\lambda_n)e_1+(-\sigma_{n-1}\lambda_{n-1}-\sigma_n\lambda_n)e_n.$$

Thus we need $L(j)=\lambda_j-\sigma_{j-1}\lambda_{j-1}\in\mathbb{Z}$ for all $j=2,\ldots,n-1$, as well as $L(1)=\lambda_1+\lambda_n\in\mathbb{Z}$ and $L(n)=-\sigma_{n-1}\lambda_{n-1}-\sigma_n\lambda_n\in\mathbb{Z}$.

Since our cycle is unbalanced, there are an odd number of $\sigma_j=-1$. Suppose, for a minute, that just one σ_j is negative; say, for instance, $\sigma_3=-1$ (Figure 3.8). Fix $\lambda\in(0,1)$, and let $\lambda_1=\lambda$. Then

$$\lambda_1=\lambda \text{ and } \lambda_2-\lambda_1\in\mathbb{Z}\Rightarrow\lambda_2=\lambda$$
$$\lambda_2=\lambda \text{ and } \lambda_3-\lambda_2\in\mathbb{Z}\Rightarrow\lambda_3=\lambda$$
$$\lambda_3=\lambda \text{ and } \lambda_4+\lambda_3\in\mathbb{Z}\Rightarrow\lambda_4=1-\lambda$$
$$\lambda_4=1-\lambda \text{ and } \lambda_5-\lambda_4\in\mathbb{Z}\Rightarrow\lambda_5=1-\lambda$$
$$\vdots$$
$$\lambda_j=1-\lambda \text{ and } \lambda_{j+1}-\lambda_j\in\mathbb{Z}\Rightarrow\lambda_{j+1}=1-\lambda$$
$$\vdots$$
$$\lambda_{n-1}=1-\lambda \text{ and } -\lambda_n-\lambda_{n-1}\in\mathbb{Z}\Rightarrow\lambda_n=\lambda$$
$$\lambda_n=\lambda \text{ and } \lambda_1+\lambda_n\in\mathbb{Z}\Rightarrow\lambda_1=1-\lambda.$$

Thus $\lambda_1 = \lambda$ and $\lambda_1 = 1 - \lambda_1$; it follows that $\lambda = \frac{1}{2}$.

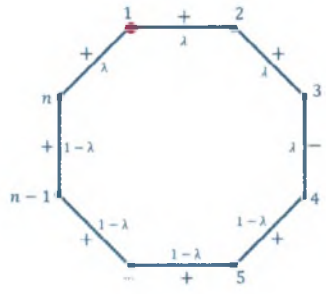

Figure 3.8: An unbalanced cycle with only one minus sign.

More generally, if $\sigma_n = 1$, the only "switches" between λ and $1 - \lambda$ occur for λ_j when $\sigma_{j-1} = -1$ — an odd number of times — as well as the two switches at λ_n and λ_1. Thus there are an odd number of switches from λ_1 to λ_1, which means that $\lambda_1 = \lambda$ and $\lambda_1 = 1 - \lambda$, which further implies that $\lambda_1 = \frac{1}{2}$.

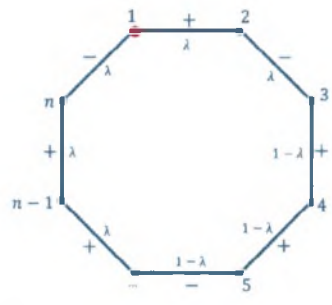

Figure 3.9: An unbalanced cycle with three minus signs, one of which is between nodes n and 1.

If $\sigma_n = -1$, we will have all the switches that occur for λ_j when $\sigma_{j-1} = -1$, which is now an even number of switches, as well as one switch at λ_1. There are,

then, an odd number of switches total, and thus $\lambda_1 = \frac{1}{2}$ once more. Figure 3.9 shows one example in which $\sigma_n = -1$.

Thus the only integer points in a cycle component occur when the non-cycle edges receive integer labels and the cycle edges receive labels from $\{0, 1, \frac{1}{2}\}$.

Thus the relative volume of $\mathcal{Z}(S)$, where G_S is a cycle component, is 2.

Let $P \subset B_d$ be a set of linearly independent vectors. By the multiplication of volume, the relative volume of $\mathcal{Z}(P)$ is $2^{cc(G_P)}$, where $cc(G_P)$ is the number of cycle components of G_P. □

3.1.3 The Ehrhart Polynomial of $\mathcal{Z}(B_d)$

Theorem 3.4. *Let*

$$\Gamma_d := \{\textit{signed graphs on } [d] \textit{ with only CC, HC, and TC}\}.$$

$\mathcal{Z}(B_d)$ is an integral zonotope, and

$$L_{\mathcal{Z}(B_d)}(t) = \sum_{G \in \Gamma_d} \left(2^{cc(G)}\right) t^{d - tc(G)}.$$

Before proving this, let's return to our original example from this section: $\mathcal{Z}(B_2)$. Figure 3.10 shows the graphs that are in Γ_2; there are six full-dimensional subsets, four 1-dimensional subsets, and one 0-dimensional subset. Only one graph contains

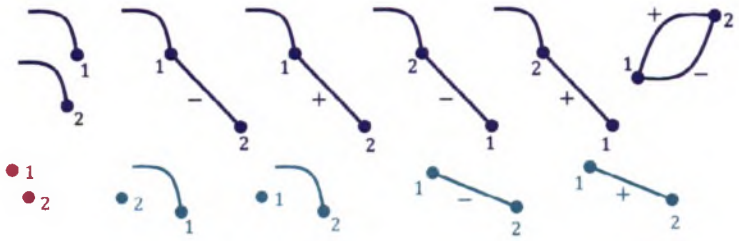

Figure 3.10: All signed graphs that correspond to linearly independent subsets of B_2.

a cycle, and it is full-dimensional. The formula in Theorem 3.4 yields

$$\begin{aligned} L_{\mathcal{Z}(B_2)}(t) &= \sum_{G \in \Gamma_2} \left(2^{cc(G)}\right) t^{2-tc(G)} \\ &= \left(2^0 t^{2-2}\right) + 4\left(2^0 t^{2-1}\right) + 5\left(2^0 t^{2-0}\right) + \left(2^1 t^{2-0}\right) \\ &= 1 + 4t + 7t^2, \end{aligned}$$

which is exactly what we got in (3.1).

Proof. By Theorem 2.13, the coefficient of t^k is the sum of the relative volumes of the k-dimensional parallelepipeds in the decomposition of $\mathcal{Z}(B_d)$. By Lemma 3.1, there is a bijection between subsets on B_d and certain signed graphs on $[d]$, and by Lemma 3.2, these subsets are linearly independent (and thus form a parallelepiped) if and only if their corresponding graph contains only CC, HC, and TC. Thus we are summing over signed graphs on $[d]$ that have only CC, HC, and TC.

Consider such a signed graph G_P on $[d]$ corresponding to a subset P of B_d. $\mathcal{Z}(P)$ is a parallelepiped, so we need to determine its dimension and relative volume.

By Lemma 3.3, the relative volume of $\mathcal{Z}(P)$ is $2^{cc(G_P)}$.

The dimension of $\mathcal{Z}(P)$ is the number of vectors in P. We turn to G_P once more. Since the edges of G_P correspond to the vectors in P, we need only count the edges of G_P. Consider each component type: CC, HC, and TC. CC and HC both have as many edges as nodes, and TC have one less edge than nodes. Since we are starting with d nodes, we can subtract 1 from that number for every TC to get the number of edges; that is, the number of edges of G_P, and thus the dimension of $\mathcal{Z}(P)$, is $d - tc(G)$. □

3.1.4 On or Off the Lattice?

What happens to the lattice point count when we translate our Coxeter permutahedra? Given our tendency to look at the parallelepipeds that tile our shapes thus far, it should come as no surprise that we look at these parallelepipeds for answers in this situation as well.

Since all of our generating vectors are integral, all these parallelepipeds also have integral vertices before being translated, regardless of which zonotopal decomposition is chosen. Therefore, since we are translating the permutahedra by a half integral vector, the vertices of the translated parallelepipeds are half integral; this can be seen for the 2-dimensional type B permutahedron in Figure 3.11.

Thus our question becomes, "What happens to the lattice point count of a parallelepiped when its vertices are half integral?" In Figure 3.11, each parallelepiped

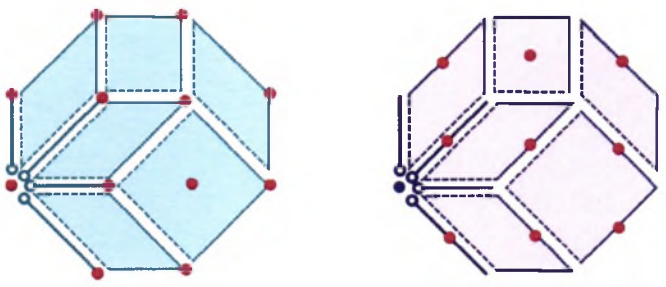

Figure 3.11: Zonotopal decompositions of $\mathcal{Z}(B_2)$ (left) and $\widetilde{\mathcal{Z}}(B_2)$ (right).

of that decomposition either keeps all of its integer points or loses all of its integer points when translated. Let $\mathcal{Z}^*(S)$ refer to a half-open parallelepiped whose generating vectors are the vectors in S.

Lemma 3.5. *Let $S \subset B_d$ be linearly independent. The number of lattice points of the $\mathcal{Z}^*(S) + \frac{1}{2}$ is*

- *the number of lattice points of the $\mathcal{Z}^*(S)$ if every TC of its corresponding graph G_S has an even number of nodes,*
- *0 otherwise.*

We call a tree component an **even TC** if it has an even number of nodes and an **odd TC** if it has an odd number of nodes.

Proof. We first consider a single tree T. In contrast to the proof of Lemma 3.3, we are looking for a labeling such that, for every node i, the sum of the labels of the edges of the form ij, ji, and i, which we call $L(i)$, is half integral. We will also be

choosing our labels from $[0, 1)$ instead of from $(0, 1]$. One thing to note is that for a tree, the number of odd degree nodes is even; thus there are an even number of even degree nodes if and only if the total number of nodes is also even.

Even TC: Let $S \subset B_d$ such that the graph corresponding to S is an even TC. We claim that $\mathcal{Z}^*(S) + \frac{1}{2}$ contains an integer point. We proceed by strong induction.

Let T be a tree with two nodes. Thus T has one edge, and we must label this edge $\frac{1}{2}$.

Suppose, then, that every tree with $2, 4, \ldots, 2n$ nodes can be labeled such that for every node, the sum of the labels of each edge going into that node is half integral. Let T be a tree on $2n + 2$ nodes. Start by finding all nodes of even degree; if there are none, we are done — we can label all edges with $\frac{1}{2}$, and since each node has odd degree, we will be adding an odd number of $\frac{1}{2}$s, which is half integral.

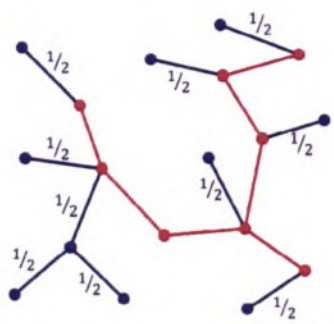

Figure 3.12: A tree $\mathcal{T}$ with an even number of nodes with the tree convex hull highlighted.

Suppose that there are at least two even degree nodes. Take the tree convex hull of these nodes; since there are at least two even degree nodes, this tree convex hull

contains at least one edge.

Every edge outside this tree convex hull must receive the label of $\frac{1}{2}$; every edge connected to a leaf l needs to be labeled with $\frac{1}{2}$ so that $L(l)$ is half integral. If there are any edges that are not yet labeled, then there is at least one node i with an even number of labeled edges—that is, an even number of edges connected to leaves. We can prune these leaves (that is, remove the leaf and its edge), since they contribute an integral value to $L(i)$. Then i becomes a new leaf, and its edge must be labeled with $\frac{1}{2}$. We can repeat this process until the only unlabeled edges are a part of the tree convex hull. Looking back at our original graph, we have now labeled every edge outside of the tree convex hull with $\frac{1}{2}$ and shown that this is the only labeling that works. An example of this labeling and tree convex hull is shown in Figure 3.12.

The tree convex hull is also a tree; this tree has at least one leaf. This leaf has an odd number of $\frac{1}{2}$s being contributed to it, since every edge outside of this tree has been labeled with a $\frac{1}{2}$. Thus this edge must be labeled with an integer value; we

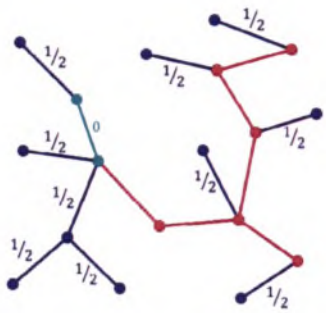

Figure 3.13: $\mathcal{T}$ with an edge labeled 0.

label it 0. Figure 3.13 shows this in the same example tree $\mathcal{T}$ used in Figure 3.12.

Removing this edge, therefore, does not influence the rest of the labels needed; let us remove this edge. Figure 3.14 shows what this looks like for $\mathcal{T}$. We are now left with two smaller trees, and if we are able to show that both of these smaller trees has an even number of nodes, we are done.

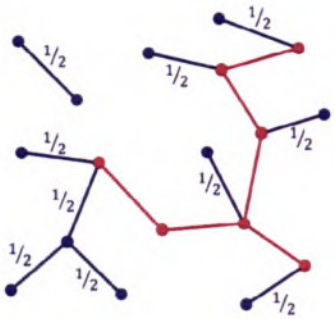

Figure 3.14: The two subtrees of $\mathcal{T}$ when the edge labeled with 0 is removed.

It suffices to show that one of the subtrees has an even number of nodes, since the total number of nodes did not change and the subtrees do not share nodes. In particular, let us consider the subtree in which every edge is already labeled with $\frac{1}{2}$. Every node in this subtree has odd degree, and as we stated at the start of our proof, the number of odd degree nodes in a tree is even. Thus this subtree has an even number of nodes, and therefore the other subtree also has an even number of nodes. Each subtree has at most $2n$ nodes and thus can be labeled in such a way that for every node i, $L(i)$ is half integral. Figure 3.15 shows the labeling for $\mathcal{T}$.

Thus a half-open parallelepiped with a corresponding graph whose TC are all

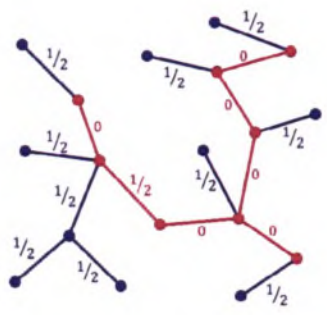

Figure 3.15: $\mathcal{T}$ labeled such that for every node i, $L(i)$ is half integral.

even TC stays on the lattice.

Odd TC: Let $S \subset B_d$ such that the graph corresponding to S is an odd TC. We claim that $\mathcal{Z}^*(S) + \frac{1}{2}$ does not contain an integer point. We shall again proceed by strong induction.

Consider a single node. Clearly there are no edge labels contributing to its sum, so the sum of the labels of the edges going into it is 0.

Consider also a tree with three nodes. We can try very hard to find a way to label the edges such that $L(i)$ is half-integral for each node i, but as soon as we label one edge $\frac{1}{2}$ to get the half integral value on one of the leaves, then we have to choose between labeling the other edge 0, to satisfy the degree two node, or $\frac{1}{2}$, to satisfy the other leaf. Examples of failed labelings are shown in Figure 3.16.

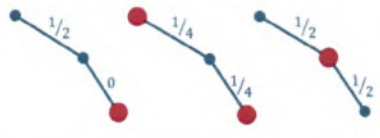

Figure 3.16: Possible labelings for a tree with 3 nodes.

Suppose, then, that every tree with $1, 3, \ldots, 2n-1$ nodes cannot be labeled in the way we want. Let T be a tree with $2n+1$ nodes. There are an odd number of even degree nodes; take their tree convex hull once more, and as in the even TC case, everything outside the tree convex hull must be labeled with $\frac{1}{2}$.

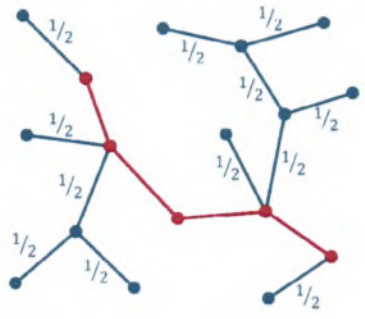

Figure 3.17: A tree $\mathcal{T}$ with an odd number of nodes and the tree convex hull highlighted.

Suppose that T has only one even degree vertex. Then we just labeled every edge in T with $\frac{1}{2}$, which means that the sum of the labels of the edges going into is even degree vertex is an even number times $\frac{1}{2}$, which is integral instead of half integral.

Suppose that T has three or more even degree vertices. Then the tree convex hull is a tree with at least two edges. As before, at least one of the nodes has degree one in the tree convex hull; we must label its edge with a 0, demonstrated in Figure 3.18.

As before, we can remove this edge without consequence, thus creating two subtrees, as shown in Figure 3.19.

Since the total number of nodes has not changed and the subtrees share no nodes,

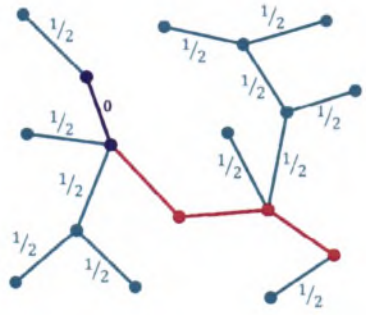

Figure 3.18: $\mathcal{T}$ with an edge labeled 0.

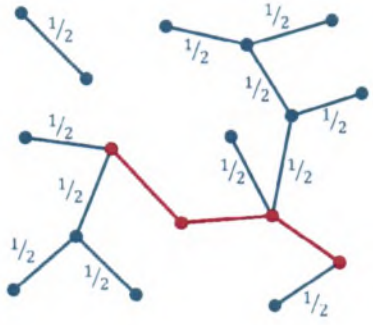

Figure 3.19: The two subtrees of $\mathcal{T}$ created by removing the edge labeled 0.

there must be one subtree with an even number of nodes and one subtree with an odd number of nodes. In particular, the subtree with an odd number of nodes has at most $2n-1$ nodes, and therefore this subtree cannot be labeled such that the sum of the edge labels for each node is half integral. Thus a half-open parallelepiped whose corresponding graph contains an odd TC is shifted off the lattice.

CC, HC, and even TC: We have one more thing to show: for $S \subset B_d$, the number of lattice points in the half-open parallelepipeds $\mathcal{Z}(S)$ and $\mathcal{Z}(S) + \frac{1}{2}$ is the

same when G_S contains only CC, HC, and even TC.

Let $S \subset B_d$ such that G_S contains only CC, HC, and even TC. Say $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$. If G_S contains only CC and HC, then $n = d$; that is, S contains d linearly independent vectors. It follows that the $\mathcal{Z}^*(S) + \frac{1}{2}$ has the same number of lattice points as the $\mathcal{Z}^*(S)$.

Suppose, then, that G_S contains at least one even TC. We just showed that the translated half-open parallelepiped that corresponds an the even TC contains an integer point; call this point p. Note that $\mathcal{Z}^*(S)$ is a fundamental parallelepiped of span $S = \{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) : \mathbf{x}, \mathbf{y} \in S, \lambda \in \mathbb{R}\}$. Note also that span S contains the origin. Let

$$\text{span } S + \frac{1}{2} = \left\{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) + \frac{1}{2} : \mathbf{x}, \mathbf{y} \in S, \lambda \in \mathbb{R}\right\};$$

$\mathcal{Z}^*(S) + \frac{1}{2}$ is a fundamental parallelepiped of span $S + \frac{1}{2}$. Then $p \in$ span $S + \frac{1}{2}$.

Our goal is to find a bijection between integer points in $\mathcal{Z}^*(S)$ and integer points in $\mathcal{Z}^*(S) + \frac{1}{2}$, thus showing that each of these half-open parallelepipeds have the same number of integer points.

Let φ : span $S \to$ span $S + \frac{1}{2}$ be the map such that $\varphi(v) = v + p$. Since p is integral, v is integral if and only if $v + p$ is integral. In particular, if $v \in \mathcal{Z}^*(S)$ is integral, then $p + v$ is an integral point in some translate of $\mathcal{Z}^*(S) + \frac{1}{2}$, so $p + v$ mod $\mathcal{Z}^*(S) + \frac{1}{2}$ is an integer point in $\mathcal{Z}^*(S) + \frac{1}{2}$. Then a point $q \in \mathcal{Z}^*(S) + \frac{1}{2}$ comes from the point $q - p \in$ span S, and $q - p$ mod $\mathcal{Z}^*(S)$ is a point in $\mathcal{Z}^*(S)$. $\square$

3.1.5 Putting It All Together

Theorem 3.6. *Let*

$$\Gamma_d := \{\textit{signed graphs on } [d] \textit{ with only CC, HC, and TC}\}$$

and

$$\widetilde{\Gamma_d} := \{\textit{signed graphs on } [d] \textit{ with only CC, HC, and even TC}\}.$$

$\widetilde{\mathcal{Z}}(B_d)$, the B_d-permutahedron centered at the origin, is a half-integral zonotope, and

$$L_{\widetilde{\mathcal{Z}}(B_d)}(t) = \begin{cases} \sum\limits_{G \in \Gamma_d} \left(2^{cc(G)}\right) t^{d-tc(G)} & \textit{if } t \textit{ even,} \\ \sum\limits_{G \in \widetilde{\Gamma_d}} \left(2^{cc(G)}\right) t^{d-tc(G)} & \textit{if } t \textit{ odd.} \end{cases}$$

Let's try this with $\widetilde{\mathcal{Z}}(B_2)$. When t is even, we have the Ehrhart polynomial for $\mathcal{Z}(B_2)$, $L_{\mathcal{Z}(B_2)} = 7t^2 + 4t + 1$. When t is odd, we only consider the graphs that have no odd TC.

As we can see from Figure 3.20, there are three graphs that have a TC that has an odd number of nodes. These are the graphs that correspond to the parallelepipeds from Figure 3.11 that lost their integer points when translated. Thus we are left with six full-dimensional subsets, two 1-dimensional subsets, and no 0-dimensional subsets. As before, only one, full-dimensional graph contains a cycle. The formula

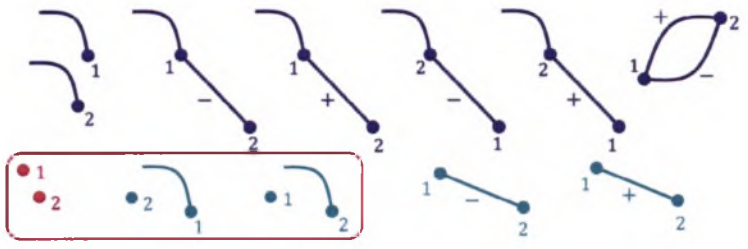

Figure 3.20: All signed graphs that correspond to subsets of B_2 with the graphs containing "odd TC" circled.

in Theorem 3.6 yields

$$\begin{aligned} L_{\mathcal{Z}(B_2)}(t) &= \sum_{G \in \widetilde{\Gamma_2}} \left(2^{cc(G)}\right) t^{2-tc(G)} \\ &= 0\left(2^0 t^{2-2}\right) + 2\left(2^0 t^{2-1}\right) + 5\left(2^0 t^{2-0}\right) + \left(2^1 t^{2-0}\right) \\ &= 2t + 7t^2. \end{aligned}$$

Thus the Ehrhart quasipolynomial for $\widetilde{\mathcal{Z}}(B_2)$ is

$$L_{\widetilde{\mathcal{Z}}(B_2)}(t) = \begin{cases} 7t^2 + 4t + 1 & \text{if } t \text{ even,} \\ 7t^2 + 2t & \text{if } t \text{ odd,} \end{cases}$$

confirming (3.2).

Proof of Theorem 3.6. If t is even, the vertices are integral; thus the even case follows from Theorem 3.4.

If t is odd, the vertices are half-integral. We want to add up the relative volumes

of all k-dimensional half-open parallelepipeds that stay on the lattice, as this will give us the coefficient of t^k. By Lemma 3.5, the only parallelepipeds that are shifted off the lattice are those that have a tree component with an odd number of nodes. Thus we are summing over signed graphs on $[d]$ that have only TC with an even number of nodes, CC, and HC.

As in the proof of Theorem 3.4, for some signed graph G_P on $[d]$ with only even TC, CC, and HC that corresponds to $P \subset B_d$, the relative volume of $\mathcal{Z}(P)$ is $2^{cc(G_P)}$ and its dimension is $d - tc(G)$. □

3.2 Type A

3.2.1 Half On, Half Off

Proposition 3.7. *The A_{d-1}-permutahedron centered at the origin, $\widetilde{\mathcal{Z}}(A_{d-1})$, is integral when d is odd and half integral when d is even.*

Proof. Recall that $A_{d-1} = \{e_i - e_j : 1 \leq i < j \leq d\}$. Since $\mathcal{Z}(A_{d-1})$ is a zonotope, $\mathcal{Z}(A_{d-1})$ is centrally symmetric. In particular, we can determine the center of mass by finding the midpoint of the line segment between two opposite vertices; we use the vertices $\mathbf{0}$ and $\sum_{1 \leq i < j \leq d} (e_i - e_j)$.

$$
\begin{aligned}
\sum_{1\le i<j\le d}(e_i-e_j) &= \begin{pmatrix}1\\-1\\0\\0\\\vdots\\0\end{pmatrix}+\cdots+\begin{pmatrix}1\\0\\0\\\vdots\\0\\-1\end{pmatrix}+\begin{pmatrix}0\\1\\-1\\0\\\vdots\\0\end{pmatrix}+\cdots+\begin{pmatrix}0\\1\\0\\\vdots\\0\\-1\end{pmatrix}+\cdots+\begin{pmatrix}0\\0\\\vdots\\0\\1\\-1\end{pmatrix}\\
&= \begin{pmatrix}d-1\\-1\\-1\\-1\\\vdots\\-1\end{pmatrix}+\begin{pmatrix}0\\d-2\\-1\\-1\\\vdots\\-1\end{pmatrix}+\begin{pmatrix}0\\0\\d-3\\-1\\\vdots\\-1\end{pmatrix}+\cdots+\begin{pmatrix}0\\0\\\vdots\\0\\1\\-1\end{pmatrix}\\
&= \begin{pmatrix}d-1\\(d-2)-1\\\vdots\\(d-i)-(i-1)\\\vdots\\1-d\end{pmatrix}\\
&= \sum_{i=1}^{d}(d-2i+1)e_i.
\end{aligned}
$$

Thus the ith entry is $d - 2i + 1$.

If d is odd, $d - 2i + 1$ is even, so $\frac{1}{2}(d - 2i + 1 - 0)$ is integral; thus the center of mass of $\mathcal{Z}(A_{d-1})$ is integral. To translate the center of mass to the origin, we therefore translate $\mathcal{Z}(A_{d-1})$ by an integral vector, thus $\widetilde{\mathcal{Z}}(A_{d-1})$ is integral.

If d is even, $d-2i+1$ is odd, so $\frac{1}{2}(d-2i+1-0)$ has denominator 2; thus the center of mass of $\mathcal{Z}(A_{d-1})$ is half-integral. To translate the center of mass to the origin, we translate $\mathcal{Z}(A_{d-1})$ by a half integral vector, thus $\widetilde{\mathcal{Z}}(A_{d-1})$ is half integral. □

3.2.2 Quasipolynomial for Type A

Note that the generators of A_{d-1} are all of the form $e_i - e_j$, $i < j$. Applying the construction of the corresponding graph G_S to a subset $S \subset A_{d-1} \subset B_d$ yields a graph with only positive edges. Thus we use unsigned graphs.

Lemma 3.8. *The subsets of A_{d-1} are in bijection with graphs on $[d]$ that have no half edges.*

Proof. Let S be a subset of A_{d-1}. Similarly to the Definition 3.1, we construct the corresponding graph G_S by adding the edge ij for each vector $e_i - e_j \in S$. □

Lemma 3.9. *Subsets of A_{d-1} are linearly independent if and only if their corresponding graph contains only TC.*

Proof. Let $S \subset A_{d-1}$ and G_S be its corresponding graph. As shown in the proof of Lemma 3.2, balanced cycles correspond to linearly dependent vectors. If we

consider S as a subset of B_d, all edges of G_S would have a positive sign, so G_S is not independent. Thus if G_S contains a cycle, S is linearly dependent. On the other hand, if G_S contains no cycles, then G_S has only TC, which are shown in the proof of Lemma 3.2 to correspond to linearly independent vectors. □

Theorem 3.10. *Let*

$$F_d := \{\textit{forests on } [d]\},$$

$$\widetilde{F_d} := \{\textit{forests on } [d] \textit{ with only even TC}\}.$$

The permutahedron $\widetilde{\mathcal{Z}}(A_{d-1})$ is a half-integral zonotope, and

$$L_{\widetilde{\mathcal{Z}}(A_{d-1})}(t) = \begin{cases} \sum\limits_{G \in F_d} t^{d-tc(G)} & \textit{if } t \textit{ even,} \\ \sum\limits_{G \in \widetilde{F_d}} t^{d-tc(G)} & \textit{if } t \textit{ odd.} \end{cases}$$

Proof. This follows from Theorem 3.4 and Lemma 3.9. □

3.3 Type D

Proposition 3.11. *The permutahedron $\widetilde{\mathcal{Z}}(D_d)$ is integral.*

Proof. Recall that $D_d = \{e_i - e_j,\ e_i + e_j : 1 \leq i < j \leq d\}$. Like with $\widetilde{\mathcal{Z}}(A_{d-1})$, we need to show that $\frac{1}{2} \sum\limits_{\mathbf{v} \in D_d} \mathbf{v}$ is integral. Since $A_{d-1} \subset D_d$, we just need to compute $\sum\limits_{1 \leq i < j \leq d} (e_i + e_j)$.

$$
\begin{aligned}
\sum_{1\le i<j\le d}(e_i+e_j) &= \begin{pmatrix}1\\1\\0\\0\\\vdots\\0\end{pmatrix}+\cdots+\begin{pmatrix}1\\0\\0\\\vdots\\0\\1\end{pmatrix}+\begin{pmatrix}0\\1\\1\\0\\\vdots\\0\end{pmatrix}+\cdots+\begin{pmatrix}0\\1\\0\\\vdots\\0\\1\end{pmatrix}+\cdots+\begin{pmatrix}0\\0\\\vdots\\0\\1\\1\end{pmatrix}\\
&= \begin{pmatrix}d-1\\1\\1\\1\\\vdots\\1\end{pmatrix}+\begin{pmatrix}0\\d-2\\1\\1\\\vdots\\1\end{pmatrix}+\begin{pmatrix}0\\0\\d-3\\1\\\vdots\\1\end{pmatrix}+\cdots+\begin{pmatrix}0\\0\\\vdots\\0\\1\\1\end{pmatrix}\\
&= \begin{pmatrix}d-1\\(d-2)+1\\\vdots\\(d-i)+(i-1)\\\vdots\\d-1\end{pmatrix}\\
&= \sum_{i=1}^{d}(d-1)e_i.
\end{aligned}
$$

Thus

$$\begin{aligned}\frac{1}{2}\sum_{\mathbf{v}\in D_d}\mathbf{v} &= \frac{1}{2}\left(\sum_{1\le i<j\le d}(e_i-e_j)+\sum_{1\le i<j\le d}(e_i+e_j)\right)\\ &= \frac{1}{2}\left(\sum_{i=1}^{d}(d-2i+1)e_i+\sum_{i=1}^{d}(d-1)e_i\right)\\ &= \frac{1}{2}\sum_{i=1}^{d}(2d-2i)e_i\\ &= \sum_{i=1}^{d}(d-i)e_i,\end{aligned}$$

which is integral. □

Thus $\widetilde{\mathcal{Z}}(D_d)$ and $\mathcal{Z}(D_d)$ have the same Ehrhart polynomial; see [1].

3.4 Type C

Theorem 3.12. *The permutahedron $\widetilde{\mathcal{Z}}(C_d)$ is integral.*

Proof. Recall that $C_d = \{e_i - e_j,\ e_i + e_j : 1 \le i < j \le d\} \cup \{2e_i : 1 \le i \le d\}$. Like with $\widetilde{\mathcal{Z}}(A_{d-1})$ and $\widetilde{\mathcal{Z}}(D_d)$, we need to show that $\frac{1}{2}\sum_{\mathbf{v}\in C_d}\mathbf{v}$ is integral. Since $D_d \subset C_d$,

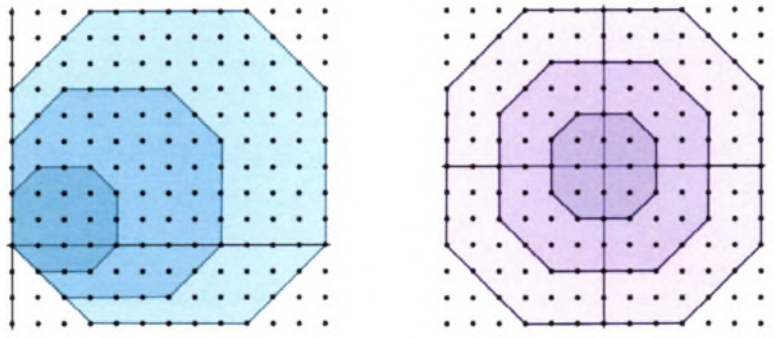

Figure 3.21: The vertices of $\widetilde{\mathcal{Z}}(C_2)$ stay on the lattice.

the only additional part is $\sum_{i=1}^{d}(2e_i)$. Thus

$$\begin{aligned}\frac{1}{2}\sum_{\mathbf{v}\in C_d}\mathbf{v} &= \frac{1}{2}\left(\sum_{1\le i<j\le d}(e_i-e_j)+\sum_{1\le i<j\le d}(e_i+e_j)+\sum_{i=1}^{d}(2e_i)\right)\\ &= \frac{1}{2}\sum_{i=1}^{d}(2d-2i+2)e_i\\ &= \sum_{i=1}^{d}(d-i+1)e_i,\end{aligned}$$

which is integral. □

Thus $\widetilde{\mathcal{Z}}(C_d)$ and $\mathcal{Z}(C_d)$ have the same Ehrhart polynomial; see [1].

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