DENSITY THEOREM FOR CONTINUOUS MULTI-WINDOW EXPONENTIALS AND APPLICATION

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CERTIFICATION OF APPROVAL

I certify that I have read *Density Theorem for continuous multi-windowed exponentials and applications* by Ben Freeman and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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In 1967 H. Landau showed a necessary condition on the density of sampling points required for reconstruction of a functions bandlimited on Ω. In this thesis, we generalize the Landau result to the continuous case providing a necessary condition for which a collection of finitely many continuous multi-windowed exponentials, is a continuous frame for $L^2(\Omega)$. We then look at various applications of this theorem, specifically in dynamical sampling.

I certify that the Abstract is a correct representation of the content of this thesis.

6/23/2018

Chair, Thesis Committee | Date

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Chapter 1

Introduction

Sampling theory is the field of mathematics concerned with reconstructing functions from various sampled points. One of the earliest examples of this is the Shannon-Nyquist sampling theorem which provides a reconstruction formula for band-limited functions (see definition 2.14) and is presented below:

$$f(x) = b\sum_{n\in\mathbb{Z}} f(bn)\frac{\sin(\pi(x-bn))}{\pi(x-bn)},$$

with the convergence of the summation uniform over compact subsets of $\mathbb{R}$. From this formula we see that f can be uniquely reconstructed from the sampling points, $f(bn)$. However in practical situations sampling points will not be uniformly distributed. In order to have a reconstruction formula we need the set to be a set of stable sampling, meaning that there exists constants $0 < A \leq B < \infty$ such that the following inequality holds for all functions f that are band-limited on a set Ω:

$$A||f||^2 \leq \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \leq B||f||^2$$

where Λ is the set of stable sampling. By the standard Fourier transform, this is equivalent to the exponential function $\{e^{2\pi i\lambda\cdot x}\}_{\lambda\in\Lambda}$, forming a frame on $L^2(\Omega)$ defined by satisfying the condition below: There exists a finite $0 < A \leq B$ such that

$$A||f||^2 \leq \sum_{\lambda\in\Lambda} |\langle f, e^{2\pi i\lambda\cdot x}\rangle|^2 \leq B||f||^2$$

for all $f \in L^2(S)$.

The classical Landau density theorem tells us that if Λ is a stable sampling set for Ω then the density is greater than $|\Omega|$ where $|\cdot|$ is the Lebesgue measure.

In this thesis we will consider the case where we have a signal f that is sampled though multi-channels by the values $\{g_i * f(\lambda_i)\}_{i=1}^N$, where g_i are smooth continuous functions, which are the channels, and λ_i is sampled through a distribution (locally finite Borel measure) μ_i. The collection $\{(g_i, \mu_i)\}_{i=1}^N$ is a convolutional stable sampling if it satisfies the following condition: There exists constants $0 < A \leq B < \infty$ such that

$$A||f||^2 \leq \sum_{i=1}^{N} \int |g_i * f(\lambda_i)|^2 d\mu_i(\lambda_i) \leq B||f||^2$$

for all band-limited functions f on Ω.

We will naturally develop the equivalence of convolutional stable sampling and continuous frames of windowed exponentials on $L^2(\Omega)$ and present a general version of the Landau density theorem under this setting

Theorem 3.1:

Let $\Omega \subseteq \mathbb{R}^d$ have finite Lebesgue measure. If $\bigcup_{j=1}^{N} \mathscr{E}(g_j, \mu_j)$ form a continuous frame for $L^2(\Omega)$, denoting $g_{\lambda,j} := g_j(x)e^{2\pi i\lambda x}$, with $||g_{\lambda,j}||_2 \leq C$ for all $j \in \{1,2,...,N\}$ and $\lambda \in supp\ \mu_j$, and denote its canonical dual as $h_{\lambda,j}$, then

$$D^-(\sum_{j=1}^{N} \mu_j) \geq \frac{|\Omega|}{C_3}$$

where $C_3 = C^2||S^{-1}||$ with S being the continuous frame operator from chapter 2. Additionally in the case of the $\mu_j's$ being the discrete measure, $C_3 = 1$.

As an application we also find a necessary sampling condition for the dynamical sampling introduced by Aldroubi et al. in [1].

The thesis is organized as follows. In Chapter 2 we will review the basics of frame theory and introduce the notion of continuous frame which will serve as the basis for the convolutional sampling and continuous frame windowed exponential. In chapter 3 we will prove our main theorem, with the applications following in the end of chapter 3.

Chapter 2

Continuous Frames and Sampling Theory

2.1 Frame Theory

In 1952, Duffin and Schaeffer introduced the notion of a Hilbert space frame in [3] when they were studying non-harmonic Fourier series. Following the work of Daubechies, Grossmann, and Meyer in [2], frame theory began to receive attention in applied harmonic analysis. Now frame theory is used widely in signal and image processing, data compression, and sampling theory.

Definition 2.1 (Frame). A sequence $\{x_n\}_{n\in I}$, where I is a countable index set, in a Hilbert space H is a *frame* for H if there exists constants $0 < A \leq B < \infty$, called the *frame bounds*, such that for all $x \in H$:

$$A||x||^2 \leq \sum_{n\in I} |\langle x, x_n \rangle|^2 \leq B||x||^2. \tag{2.1}$$

When $A = B$, we say that $\{x_n\}_{n\in I}$ is a *tight frame*. Likewise when $A = B = 1$ we say that $\{x_n\}_{n\in I}$ is a *Parseval frame*.

Frames should be regarded as an overcomplete basis, one where in general a vector $\mathbf{x} \in H$ can be expanded in a nonunique way. Because of such redundancy, frames are widely used in real-world applications like signal processing. If H is finite-dimensional, we note that any spanning set of H is a frame. For more details concerning frame theory we refer the reader to [5].

Example 2.1. Let $H = \mathbb{R}^2$, and $\{x_n\}_{n\in I} = \{(0,1), (\frac{-\sqrt{3}}{2}, \frac{-1}{2}), (\frac{\sqrt{3}}{2}, \frac{-1}{2})\}$. We see that $\{x_n\}_{n\in I}$ spans $\mathbb{R}^2$ and satisfies the equation:

$$\sum_{n\in I} | \langle x, x_n \rangle |^2 = \frac{3}{2} ||x||^2.$$

This frame $\{x_n\}_{n\in I}$ is commonly referred to as the Mercedes-Benz frame.

Definition 2.2 (Exact Frame). A frame, $\{x_n\}$ is an *exact frame* when the subset $\{x_n\}_{n\neq j}$ ceases to be a frame for any j.

An exact frame is also called a *Riesz basis*. In addition to the different types of frames listed above, we now to introduce the three important operators in frame theory.

Definition 2.3 (Analysis Operator). Given a Hilbert space H and $f \in H$ with $\{x_n\}_{n\in I}$ a frame in H then the *analysis operator* of $\{x_n\}_{n\in I}$ is defined as $T : H \to \ell^2(I)$ by $f \mapsto \{\langle f, x_n \rangle\}_{n\in I}$.

Definition 2.4 (Synthesis Operator)**.** Given a Hilbert space H and $f \in H$ with $\{x_n\}_{n\in I}$ a frame in H then the *synthesis operator* of $\{x_n\}_{n\in I}$ is defined as $T^* : \ell^2(I) \to H$ by $a_n \mapsto \sum_{n\in I} a_n x_n$.

Finally we define the composition of the analysis operator and the synthesis operator, called the frame operator.

Definition 2.5 (Frame Operator)**.** Given a Hilbert space H and $f \in H$ with $\{x_n\}_{n\in I}$ a frame in H then the *frame operator* of $\{x_n\}_{n\in I}$ is defined as $S := T^*T : H \to H$ by $Sf = \sum_{n\in I} \langle f, x_n \rangle x_n$.

Note the frame inequality, (2.1) is equivalent to the following:

$$A||f||^2 \leq \langle Sf, f \rangle \leq B||f||^2 \iff AI \leq S \leq BI.$$

One can then prove that S is both self-adjoint and invertible as in [5]. Therefore we see that:

$$f = SS^{-1}f = \sum_{n\in I} \langle S^{-1}f, x_n \rangle x_n = \sum_{n\in I} \langle f, S^{-1}x_n \rangle x_n.$$

This provides a reconstruction formula for f and motivates our next definition.

Definition 2.6 (Canonical Dual Frame)**.** Given a Hilbert space H and $f \in H$ with $\{x_n\}_{n\in I}$ a frame in H then $\{S^{-1}x_n\}_{n\in I}$ forms the *canonical dual frame* of $\{x_n\}_{n\in I}$.

In general $\{S^{-1}x_n\}_{n\in I}$ is not the only frame that provides a reconstruction formula, giving rise to the notion of a dual frame.

Definition 2.7 (Dual Frame)**.** Given a Hilbert space H with $\{x_n\}_{n\in I}$ a frame in H then any frame $\{y_n\}_{n\in I}$ in H that satisfies the following equation is called a *dual frame* of $\{x_n\}_{n\in I}$:

$$f = \sum_{n=1}^{N} \langle f, y_n \rangle x_n$$

for all $f \in H$.

2.2 Continuous Frames

Definition 2.8 (Continuous Frame)**.** Let H be a Hilbert space, X a locally compact space, and μ a locally finite Borel measure on X. Then a set of vectors in H $\{f_x\}_{x\in X}$ along with a measure μ is said to be a *continuous frame* if there exists constants $0 < A \leq B < \infty$ such that

$$A||f||^2 \leq \int_X |\langle f, f_x \rangle|^2 d\mu(x) \leq B||f||^2 \tag{2.2}$$

for all $f \in H$ and the map $x \mapsto \langle f, f_x \rangle$ is μ-measurable for every f.

Example 2.2. In the case that $X = \mathbb{N}$, and $\mu = \delta_{\mathbb{N}}$ we see that

$$A||f||^2 \leq \int_X |\langle f, f_x \rangle|^2 d\mu(x) \leq B||f||^2$$

reduces to

$$A||f||^2 \leq \sum_{x=1}^{\infty} |\langle f, f_x \rangle|^2 \leq B||f||^2.$$

Therefore the continuous frame is a natural generalization of the frames mentioned earlier in this chapter.

To define the analysis, synthesis, and frame operators in the continuous frame setting analogously to before, we need the following weak formulation of an integral of Hilbert-space valued functions.

Definition 2.9. Integrals of the form

$$\int f_x d\mu(x), \quad f_x : X \mapsto H$$

are defined in the following sense: $\int f_x d\mu(x)$ is defined to be the unique element $C \in H$ such that

$$\langle y, C\rangle = \int \langle y, f_x\rangle \, d\mu(x) \;\; \forall y \in H.$$

Definition 2.10 (Continuous Analysis Operator)**.** Given a continuous frame $\{f_x\}_{x\in X}$ the *continuous analysis operator* $T : H \to L^2(X,\mu)$ is defined by:
$f \mapsto \langle f, f_x\rangle_{x\in X}$

Definition 2.11 (Continuous Synthesis Operator)**.** Given a continuous frame $\{f_x\}_{x\in X}$ the *continuous synthesis operator* $T^* : L^2(H) \to H$ is defined by:
$\langle f, f_x\rangle \mapsto \int_X \langle f, f_x\rangle \, d\mu(x)$

Definition 2.12 (Continuous Frame Operator)**.** Given a continuous frame $\{f_x\}_{x\in X}$ the *continuous frame operator* is defined as follows using the weak formulation:
$S : H \to H$ by $Sf := \int_X \langle f, f_x\rangle \, f_x d\mu(x)$

Definition 2.13 (Continuous Dual Frame). Given a continuous frame $\{f_x\}_{x\in X}$, another continuous frame $\{g_x\}_{x\in X}$ is said to be a *continuous dual frame* of $\{f_x\}_{x\in X}$ if it satisfies the following condition for all $f \in H$:

$$f = \int_X \langle f, f_x \rangle g_x d\mu(x).$$

In particular $\{S^{-1}f_x\}_{x\in X}$ is a continuous dual frame of $\{f_x\}_{x\in X}$ and is called the canonical continuous dual frame.

2.3 Sampling and Windowed Exponentials

In this section, we will study the relationship between sampling theory, frames, and windowed exponentials. Detailed exposition can be found in [9] and [5]. This relationship will form the foundation for our main theorem in the following chapter.

2.3.1 Sampling Theory

Given a signal (function) f and a discrete set of samples of f, $\{f(x_n)\}_{n\in I}$ a natural question is whether or not the samples can be used to reconstruct the original signal. In order to obtain the reconstruction we impose the natural assumption that f is in the Paley–Wiener Space.

Definition 2.14 (Paley–Wiener Space). Let $\Omega \in \mathbb{R}^d$ be a measurable set of finite Lebesgue measure. The *Paley–Wiener space* of functions for a set Ω, denoted PW_Ω, is the set of all

$L^2(\mathbb{R}^d)$ functions that have compactly supported Fourier transforms in Ω

$$PW_\Omega := \left\{ f \in L^2(\mathbb{R}^d) : supp(\hat{f}) \subseteq \Omega \right\}.$$

If $f \in PW_\Omega$, f is typically referred to as a band-limited function.

One result of the reconstruction of band-limited functions is the following theorem:

Theorem 2.1 (Shannon Sampling Theorem). *If $0 < b \leq 1$ then for all $f \in PW_{[-b,b]}$ we have*

$$f(x) = b \sum_{n \in \mathbb{Z}} f(bn) \frac{\sin(\pi(x-bn))}{\pi(x-bn)}$$

Where the sum converges uniformly over compact sets.

In general we define sampling in the following sense:

Definition 2.15 (Stable Sampling). Let g be a function, then a set Λ is said to be ag *convolution stable sampling* set for f if there exists constants $0 < A \leq B < \infty$ such that:

$$A||f||^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B||f||^2, \ \forall f \in PW_\Omega$$

By the Parseval and Plancherel theorems, one can deduce the following theorem:

Theorem 2.2. *Λ is a stable sampling set for PW_Ω if and only if $\{e^{2\pi i \lambda x}\}_{\lambda \in \Lambda}$ is a frame for $L^2(\Omega)$.*

2.3.2 Windowed Exponentials and Convolutional Sampling

Samples of band-limited functions can also be taken through convolutions with certain kernels.

Definition 2.16 (g convoluational stable sampling). Let g be a function, then a set Λ is said to be a g *convolution stable sampling* set for f if there exists constants $0 < A \leq B < \infty$ such that:

$$A||f||^2 \leq \sum_{\lambda\in\Lambda} |g * f(\lambda)|^2 \leq B||f||^2.$$

Definition 2.17 (Windowed Exponential). Given a Hilbert space H, a set $\Lambda \subset H$ and a function g, the *windowed exponential* of g is the set $\{g(x)e^{2\pi i\langle\lambda,x\rangle}\}_{\lambda\in\Lambda} =: \mathscr{E}(g,\Lambda)$.

We can now provide the following generalization of Theorem 2.2 :

Theorem 2.3. *The following are equivalent:*

1. $\mathscr{E}(g,\Lambda)$ *is a frame for* $L^2(\Omega)$.
2. *A uniformly discrete set* Λ *is a* $\overline{G}$ *convolution stable sampling set for* PW_Ω*, where* $G = \widehat{(g\ 1_\Omega)}$ *and* 1_Ω *is the indicator function of* Ω
3. $\{G(\cdot - \lambda)\}_{\lambda\in\Lambda}$ *is a frame for* PW_Ω.

Proof. Let $\mathscr{E}(\Lambda,g)$ be a frame for $L^2(\Omega)$. Then there exists $0 < A \leq B < \infty$ such that for all $f \in L^2(\Omega)$:

$$A\int_\Omega |f|^2 \leq \sum_{\lambda\in\Lambda} |\int f(x)\left(\overline{g(x)e^{2\pi i\lambda x}1_\Omega(x)}\right) dx|^2 \leq B\int_\Omega |f|^2.$$

Looking at the value inside the middle absolute value we make the following observations:

$$\int f(x)\left(\overline{g(x)e^{2\pi i\lambda x}1_\Omega(x)}\right)dx = \int \hat{f}(\xi)\overline{(ge^{2\pi i\langle\lambda,\cdot\rangle}1_\Omega)\widehat{\ }(\xi)}d\xi \tag{2.3}$$

$$= \int \hat{f}(\xi)\overline{(g\,1_\Omega)\widehat{\ }(\xi-\lambda)}d\xi. \tag{2.4}$$

$$= \langle \hat{f}, (g\,1_\Omega)\widehat{\ }(\cdot-\lambda)\rangle \tag{2.5}$$

$$= \overline{(g\,1_\Omega)\widehat{\ }} * \hat{f}. \tag{2.6}$$

By replacing the quantity on the left-hand side of (2.3) with the quantity in (2.5) and noting that $\langle \hat{f}, \widehat{(g\,1_\Omega)}(\cdot-\lambda)\rangle = \langle f, (g\,1_\Omega)\rangle$ by the Plancherel theorem, we see that 1 implies 2. Likewise we note that the equality of lines 2.3 through 2.6 shows that 2 implies 3 and 3 implies 1, completing the theorem.

□

The main problem now when thinking about sampling is to classify which (Λ, g) will generate a convolutional stable sampling or equivalently a windowed exponential. One of the well-known results by H. Landau [7] states that the necessary lower Beurling Density, $D^-(\Lambda)$, is at least as much as the Lebesgue measure of the set Ω.

Definition 2.18. Denote a cube of side length k and center x as $Q_k(x)$.

Definition 2.19 (Lower Beurling Density)**.** The *lower Beurling density* of a set $\Lambda \subset \mathbb{R}^d$,

$D^-(\Lambda)$, is defined as follows:

$$D^-(\Lambda) = \liminf_{r\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_r(x))}{r^d}.$$

Theorem 2.4 (Landau Density Theorem). *Let $\Omega, \Lambda \subseteq \mathbb{R}^d$. Then Λ is a stable sampling set for Ω if and only if:*

$$D^-(\Lambda) \geq |\Omega|.$$

2.3.3 Multi-windowed Exponentials and Multi-Channel Convolutional Sampling

In this section we also define multi-windowed exponentials and multi-channel convolutional sampling.

Definition 2.20 (Multi-Windowed Exponentials). Given H a Hilbert space, a collection of sets $\Lambda_j \subset H$, with $1 \leq j \leq N$ and a set of functions $\{g_j\}_{j=1}^N$ the *multi-windowed exponential* of $\{g_j\}$ is $\mathscr{E} = \bigcup_{j=1}^{N} \mathscr{E}(g_j, \Lambda_j)$.

Definition 2.21 (Multi-Channel Convolutional Sampling). A collection of sets Λ_j is said to be a *multi-channel g_j convolution stable sampling* set for f if there exists constants $0 < A \leq B < \infty$ such that:

$$A||f||^2 \leq \sum_{j=1}^{N} \sum_{\lambda\in\Lambda_j} |g_j * f(\lambda)|^2 \leq B||f||^2.$$

We now generalize the concept of a windowed exponential into the continuous setting.

Definition 2.22 (Continuous Multi-Windowed Exponential)**.** Given a Hilbert space H,a set of measures $\{\mu_j\}_{j=1}^N$ and a set of functions $\{g_j\}_{j=1}^N$, we denote the *continuous multi-windowed exponential* of g_j and μ_j as $\bigcup_{j=1}^{N}\mathscr{E}(g_j,\mu_j)$. A *continuous multi-windowed exponential* is a frame for Ω if, given a set $\Omega\in\mathbb{R}^d$ and collection of L^2 functions $\{g_j\}_{j=1}^N$, there exists constants $0<A\leq B<\infty$ such that for all $f\in L^2(\Omega)$:

$$A||f||^2\leq\sum_{j=1}^{N}\int_{\mathbb{R}^d}\left|\int_\Omega f(x)\overline{g}_j(x)e^{2\pi i\langle\lambda,x\rangle}dx\right|^2 d\mu_j(\lambda)\leq B||f||^2.$$

Example 2.3. Let m be the Lebesgue measure of $\mathbb{R}^d$ and let $\Omega\subseteq\mathbb{R}^d$, then $\mathscr{E}(1_\Omega,m)$ forms a continuous frame over $L^2(\Omega)$

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\int_\Omega f(x)e^{-2\pi i\lambda x}dx\right|^2 dm(\lambda) &= \int|(f1_\Omega)\hat{}(\lambda)|^2 d\lambda\\ &= \int|f1_\Omega(x)|^2dx\\ &= \int_\Omega|f|^2dx,\end{aligned}$$

where the first and second lines are equal by the Plancherel identity.

The critical difference between the continuous windowed exponential and windowed exponentials is that the λ which are elements of a set $\Lambda\in\mathbb{R}^d$ for windowed exponentials are instead only required to be from $\mathbb{R}^d$ according to the distribution μ_j for continuous windowed exponentials. The continuous analog of the Beurling density is defined below.

Definition 2.23 (Continuous Lower Beurling Density). The *continuous lower Beurling density* of a measure, μ, of a set $\mathbb{R}^d$ is defined as

$$D^-(\mu) = \liminf_{r \to \infty} \inf_x \frac{\mu(Q_r(x))}{r^d}.$$

In the next chapter we will prove our main theorem for the necessary density condition for the frame of continuous multi-windowed exponentials.

Chapter 3

Generalized Landau Density Theorem

In this chapter we will prove a generalization of the Landau density theorem in the setting of continuous multi-windowed exponentials. As mentioned in the previous section, H. Landau determined the minimum density required for stable sampling in his 1967 paper [7]. The particular manner in which we prove this generalization is adapted from a simplified proof of the original Landau density theorem by S. Nitzan and A. Olevskii in [8]. Using the notation from the previous chapter we can now state our main result, the generalized Landau density theorem:

Theorem 3.1. *Let $\Omega \in \mathbb{R}^d$ have finite Lebesgue measure. If $\bigcup_{j=1}^{N} \mathscr{E}(g_j, \mu_j)$ form a continuous frame for $L^2(\Omega)$, denoting $g_{\lambda,j} := g_j(x)e^{2\pi i\lambda x}$, with $||g_{\lambda,j}||_2 \leq C$ for all $j \in \{1,2,...,N\}$ and $\lambda \in$ supp μ_j, and denote its canonical dual as $h_{\lambda,j}$, then*

$$D^-(\sum_{j=1}^{N}\mu_j) \geq \frac{|\Omega|}{C_3}$$

where $C_3 = C^2||S^{-1}||$.

Note that when $N = 1$ and μ is the counting measure, we retrieve Landau's classical result. We also note that the inverse Fourier image of $\bigcup_{j=1}^{N}(g_j, \mu_j)$ is $\bigcup_{j=1}^{N}(\check{g}_j, \mu_j)$ which forms a frame for PW_Ω by Theorem 2.3. The dual frame of $\bigcup_{j=1}^{N}(\check{g}_j, \mu_j)$ is the inverse Fourier image of $\mathscr{E}$, which we will denote by $\check{\mathscr{E}}$.

3.0.1 Lemmas

In this section we will state some definitions and lemmas necessary for the proof of the main theorem.

Lemma 3.2. *Let* (v_λ, μ) *and* (u_λ, μ) *be continuous dual frames in* $L^2(\Omega)$ *then,*

$$\int \check{v}_\lambda(x)\overline{\check{u}_\lambda(x)}d\mu(\lambda) = |\Omega|,$$

where $\check{v}_\lambda, \check{u}_\lambda$ *are the inverse Fourier transforms of* v_λ *and* u_λ *respectively, ie.*

$$\check{v}_\lambda = \langle v_\lambda, e_{-x}\rangle\,.$$

Proof. Using the definition of continuous dual frame in the second last line we have

$$\begin{aligned}
&\int \check{v}_\lambda(x)\overline{\check{u}_\lambda(x)}d\mu(\lambda) \\
&= \int \langle v_\lambda, e_{-x}\rangle \langle e_{-x}, u_\lambda\rangle d\mu(\lambda) \\
&= \langle \int \langle e_{-x}, u_\lambda\rangle v_\lambda d\mu(\lambda), e_{-x}\rangle \\
&= \langle e_{-x}, e_{-x}\rangle = |\Omega|.
\end{aligned}$$

□

Definition 3.1 (Wiener–amalgam Space). Given $1 \le p,\ q \le \infty$ and $\alpha > 0$ the *Wiener–amalgam space*, $W(L^p, \ell^q)$ is all of the functions, f on $\mathbb{R}^d$ such that

$$\left(\sum_{k\in\mathbb{Z}^d} ||f \cdot \chi_{Q_\alpha(\alpha k)}||_p^q\right)^{\frac{1}{q}} < \infty.$$

The following lemma is proposition 2.12 from [6].

Lemma 3.3. *Let Ω be a bounded subset of $\mathbb{R}^d$, and let $1 \le p \le 2$ be given. Then for $f \in L^2(\Omega)$ we have*

$$\hat{f} \in L^p(\mathbb{R}^d) \implies \hat{f}. \in W(C, \ell^p) = \{F \in W(L^\infty, \ell^q) : F \textit{ is Continuous}\}.$$

Proof. Proof in [6]. □

Lemma 3.4. $\check{\mathscr{E}}$ *has the following properties:*

1. $||\check{h}_{\lambda,j}||_2 \leq C_1$, *for some* C_1 *and* $\forall\, \check{h}_{\lambda,j} \in \check{\mathscr{E}}$.

2. $\sum_{j=1}^{N} \int |\check{h}_{\lambda,j}(x)|^2 d\mu_j(\lambda) \leq C_2$, *where* C_2 *is the upper frame bound of* $\mathscr{E}$ *and* $\forall x$.

3. $|\langle \check{g}_{\lambda,j}, \check{h}_{\lambda,j}\rangle| \leq C_3\ \forall j \in \{1,2,...,N\}$ *and* $\forall \lambda \in supp\ \mu_j$.

Proof. By Plancherel's theorem we know that $||\check{h}_{\lambda,j}||_2 = ||h_{\lambda,j}||_2$. Additionally we note that

$$\begin{aligned}||h_{\lambda,j}||_2 &\leq ||S^{-1}|| \cdot ||g_{\lambda,j}||_2\\ &\leq ||S^{-1}|| \cdot ||g_j||_2\\ &\leq ||S^{-1}|| \underset{j}{Max} ||g_j||_2 = C_1\end{aligned}$$

justifying claim 1. Claim 2 follows from the fact that $\mathscr{E}$ is a frame in $L^2(\Omega)$:

$$\sum_{j=1}^{N} \int |\check{h}_{\lambda,j}(x)|^2 d\mu_j(\lambda) = \sum_{j=1}^{N} \int |\langle h_{\lambda,j}, e_x\rangle|^2 d\mu_j(\lambda) \leq C_2 ||e_x||^2_{L^2(\Omega)}.$$

For claim 3 we note by the Cauchy–Schwartz inequality

$$\begin{aligned}|\langle \check{g}_{\lambda,j}, \check{h}_{\lambda,j}\rangle| &\leq ||\check{g}_{\lambda,j}||_2 \cdot ||\check{h}_{\lambda,j}||_2\\ &\leq ||\check{g}_{\lambda,j}||_2 \cdot C_1\\ &\leq \underset{j}{Max} ||g_j||_2 \cdot C_1\\ &= C_3.\end{aligned}$$

□

Finally we note that by Proposition 17 in [4] that we have the following lemma:

Lemma 3.5. *If* $\mathscr{E} = \bigcup_{j=1}^{N} \mathscr{E}(g_j, \mu_j)$ *forms a frame then* μ_j *has the property that there exists a* $C_j > 0$ *such that:*

$$\sup_{x \in \mathbb{R}^d} \mu_j(Q_1(x)) \leq C_j.$$

Measures with this property are called translation bounded measures.

3.0.2 Proof of the Main Theorem

Let $\varepsilon > 0$. By Lemma 3.3 $g_{\lambda,j} \in W(C, l^1)$ thus we can find a b large enough such that

$$\sum_{n \in \mathbb{Z}^d \setminus Q_b(0)} \sup_{\xi \in Q_1(n)} |\check{g}_j(\xi)|^2 < \varepsilon^2. \tag{3.1}$$

We claim that

$$\lim_{R \to \infty} \sup_{x \in \mathbb{R}^d} \sum_{j=1}^{N} \int_{(\lambda - x) \in Q_R(x)^c} |\check{g}_j(\lambda - x)|^2 d\mu_j(\lambda) = 0. \tag{3.2}$$

Proof. For any $R > b$ we have the following:

$$\begin{aligned}
&\sum_{j=1}^{N}\int_{(\lambda-x)\in Q_R(x)^c}|\check{g}_j(\lambda-x)|^2 d\mu_j(\lambda)\\
\leq&\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}^d\setminus Q_b(0)}\int_{(\lambda-x)\in Q_1(n)}|\check{g}_j(\lambda-x)|^2 d\mu_j(\lambda)\\
\leq&\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}^d\setminus Q_b(0)}\int_{(\lambda-x)\in Q_1(n)}\sup_{\xi\in Q_1(n)}|\check{g}_j(\xi)|^2 d\mu_j(\lambda)\\
\leq&\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}^d\setminus Q_b(0)}\mu_j(Q_1(n))\sup_{\xi\in Q_1(n)}|\check{g}_j(\xi)|^2\\
\leq&\sum_{j=1}^{N}\sum_{n\in\mathbb{Z}^d\setminus Q_b(0)}C_j\sup_{\xi\in Q_1(n)}|\check{g}_j(\xi)|^2\\
\leq&\sum_{j=1}^{N}C_j\sum_{n\in\mathbb{Z}^d\setminus Q_b(0)}\sup_{\xi\in Q_1(n)}|\check{g}_j(\xi)|^2\\
\leq&\sum_{j=1}^{N}C_j\varepsilon^2,
\end{aligned}$$

where the third to last inequality is due to Lemma 3.5. □

Therefore we have the conclusion that

$$\sum_{j=1}^{N}\int_{(\lambda-x)\in B_R(x)^c}|\check{g}_j(\lambda-x)|^2 d\mu_j(\lambda) < \varepsilon^2 \quad \forall x\in\mathbb{R}^d \tag{3.3}$$

for any sufficiently large R. Now consider a cube $Q_r(y)$ with $r > 2b$.

Let $Q^+ := Q_r(y) + Q_{2b}(0)$ and let Q^- be the cube such that $Q^- + Q_{2b}(0) = Q_r(y)$. By Lemma 3.2 we see that

$$A := \sum_{j=1}^{N} \int_{\mathbb{R}^d} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu(\lambda) = |\Omega|.$$

Next we separate A into:

- $A_1 := \sum\limits_{j=1}^{N} \int\limits_{Q^-} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu_j(\lambda)$
- $A_2 := \sum\limits_{j=1}^{N} \int\limits_{Q^+\setminus Q^-} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu_j(\lambda)$
- $A_3 := \sum\limits_{j=1}^{N} \int\limits_{\mathbb{R}\setminus Q^+} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu_j(\lambda).$

We can see with claim 2 of Lemma 3.4, (3.3) and the Cauchy–Schwartz inequality that

$$\begin{aligned}
\int_{Q_r(y)} |A_3|dx &= \int_{Q_r(y)} |\sum_{j=1}^{N} \int_{\mathbb{R}^d\setminus Q^+} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu_j(\lambda)|dx \\
&\leq \int_{Q_r(y)} \sum_{j=1}^{N} |\int_{\mathbb{R}^d\setminus Q^+} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu_j(\lambda)|dx \\
&\leq \int_{Q_r(y)} \sum_{j=1}^{N} \int_{\mathbb{R}^d\setminus Q^+} |\check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}|d\mu_j(\lambda)dx \\
&\leq \int_{Q_r(y)} C_2 \cdot \varepsilon dx \\
&= C_2|Q_r(y)|\varepsilon.
\end{aligned}$$

Next from claim 3 of Lemma 3.4, (3.3) and the Cauchy–Schwartz inequality we have

$$\begin{aligned}\left|\int_{Q_r(y)} A_1\right| &= \left|\int_{Q_r(y)} \sum_{j=1}^{N} \int_{Q^-} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu_j(\lambda)dx\right| \\ &= \left|\sum_{j=1}^{N} \int_{Q^-} \int_{Q_r(y)} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}dxd\mu_j(\lambda)\right| \\ &= \left|\sum_{j=1}^{N} \int_{Q^-} \left(\langle \check{g}_{\lambda,j}, \check{h}_{\lambda,j}\rangle - \int_{\mathbb{R}^d\setminus Q_r(y)} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}dx\right) d\mu_j(\lambda)\right| \\ &\leq \sum_{j=1}^{N} \int_{Q^-} |\langle \check{g}_{\lambda,j}, \check{h}_{\lambda,j}\rangle| d\mu_j(\lambda) + \sum_{j=1}^{N} \int_{Q^-} \left|\int_{\mathbb{R}^d\setminus Q_r(y)} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}dx\right| d\mu_j(\lambda) \\ &\leq \sum_{j=1}^{N} \int_{Q^-} C_3\, d\mu_j(\lambda) + \sum_{j=1}^{N} \int_{Q^-} \left|\int_{\mathbb{R}^d\setminus Q_r(y)} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}dx\right| d\mu_j(\lambda) \\ &\leq C_3 \sum_{j=1}^{N} \mu_j(Q_r(y)) + C_2\varepsilon \sum_{j=1}^{N} \mu_j(Q_r(y)).\end{aligned}$$

From claim 1 of Lemma 3.4, the Cauchy–Schwartz inequality, and the condition that $||g_{\lambda,j}||_2 \leq C$ for all $j \in \{1,2,...,N\}$ and $\lambda \in \text{supp}\ \mu_j$ we obtain:

$$\begin{aligned}\int_{Q_r(y)} |A_2|dx &= \int_{Q_r(y)} |\sum_{j=1}^{N} \int_{Q^+\setminus Q^-} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu_j(\lambda)|dx \\ &\leq \int_{Q_r(y)} \sum_{j=1}^{N} |\int_{Q^+\setminus Q^-} \check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}d\mu_j(\lambda)|dx \\ &\leq \int_{Q^+\setminus Q^-} \sum_{j=1}^{N} \int_{Q_r(y)} |\check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}|dxd\mu_j(\lambda)\end{aligned}$$

$$\leq \int_{Q^+\setminus Q^-} \sum_{j=1}^{N} \int_{\mathbb{R}^d} |\check{g}_{\lambda,j}(x)\overline{\check{h}_{\lambda,j}(x)}|dxd\mu_j(\lambda)$$
$$\leq \int_{Q^+\setminus Q^-} \sum_{j=1}^{N} \left((||\check{g}_{\lambda,j}||_2)^{\frac{1}{2}} (||\check{h}_{\lambda,j}||_2)^{\frac{1}{2}} \right) d\mu_j(\lambda)$$
$$\leq \int_{Q^+\setminus Q^-} \sum_{j=1}^{N} C^{\frac{1}{2}} \cdot C_1^{\frac{1}{2}} d\mu_j(\lambda)$$
$$= C^{\frac{1}{2}} \cdot C_1^{\frac{1}{2}} \sum_{j=1}^{N} \mu_j(Q^+ \setminus Q^-).$$

Combining all of this we obtain our desired result:

$$|Q_r(y)||\Omega| = \int_{Q_r(y)} Adx$$
$$\leq C_3 \sum_{j=1}^{N} \mu_j(Q_r(y)) + C_2\varepsilon \sum_{j=1}^{N} \mu_j(Q_r(y)) + C^{\frac{1}{2}} \cdot C_1^{\frac{1}{2}} \sum_{j=1}^{N} \mu_j(Q^+ \setminus Q^-) + C_2|Q_r(y)|\varepsilon.$$

Thus,

$$\frac{C_3 \sum_{j=1}^{N} \mu_j(Q_r(y) + C_2\varepsilon \sum_{j=1}^{N} \mu_j(Q_r(y))}{|Q_r(y)|} \geq |\Omega| - \frac{C^{\frac{1}{2}} \cdot C_1^{\frac{1}{2}}}{|Q_r(y)|} \sum_{j=1}^{N} \mu_j(Q^+ \setminus Q^-) - C_2\varepsilon.$$

Therefore,

$$\min_{r=|Q_r(y)|, r>2b} \frac{C_3 \sum_{j=1}^{N} \mu_j(Q_r(y)) + C_2\varepsilon \sum_{j=1}^{N} \mu_j(Q_r(y))}{r^d} > |\Omega| - \frac{C^{\frac{1}{2}} \cdot C_1^{\frac{1}{2}}}{r^d} \sum_{j=1}^{N} \mu_j(Q^+ \setminus Q^-) - C_2\varepsilon.$$

Now if we let $r \to \infty$ and $\varepsilon \to 0$ and we note that $\mu_j(Q^+ \setminus Q^-)$ is bounded because $Q^+ \setminus Q^-$

is a square annulus of width $2b$ we see that

$$C_3 \cdot D^- \left(\sum_{j=1}^{N} \mu_j \right) = \liminf_{r\to\infty} \inf_{r=|Q_r(y)|} \frac{C_3 \sum\limits_{j=1}^{N} \mu_j(Q_r(y))}{r^d} \geq |\Omega|$$

as desired.

3.0.3 Applications

Corollary 3.6. *Let* $\{\Lambda_j\}_{j=1}^N$ *be a collection of discrete sets and* $\mu_j = \delta_{\Lambda_j}$. *Then* $C_3 = 1$ *and* $D^-(\sum\limits_{j=1}^{N} \mu_j) \geq |\Omega|$.

Proof. Note that the C_3 constant emerges from the term $|\langle \breve{g}_{\lambda,j}, \check{h}_{\lambda,j}\rangle|$ which in the discrete case is bounded by 1. This is due to the fact from [5] that for any frame $\{x_n\}$ if $x = \sum\limits_{n=1}^{N} c_n x_n$ then

$$\sum_{n=1}^{N} |c_n|^2 = \sum_{n=1}^{N} |\langle x, S^{-1}x_n\rangle|^2 + \sum_{n=1}^{N} |c_n - \langle x, S^{-1}x_n\rangle|^2.$$

Thus in the specific case that $x = x_n$ we see that $c_i = 0$ for all $i \neq n$ and $c_n = 1$. Hence

$$1 = \sum_{i=1}^{N} |\langle x_n, S^{-1}x_i\rangle|^2 + \sum_{i=1}^{N} |\delta_n(i) - \langle x_n, S^{-1}x_i\rangle|^2.$$

Therefore we see that $1 \geq \sum\limits_{i=1}^{N} |\langle x_n, S^{-1}x_i\rangle|^2$ and specifically $1 \geq |\langle x_n, S^{-1}x_i\rangle|^2$. Since $\check{h}_{\lambda,j} = S^{-1}\breve{g}_{\lambda,j}$ we see that this result implies $1 \geq |\langle \breve{g}_{\lambda,j}, \check{h}_{\lambda,j}\rangle|^2$ as desired. Next we note that in the case that $\mu_j = \delta_{\Lambda_j}$

$$\liminf_{r\to\infty}\inf_{r=|Q_r(y)|}\frac{C_3\sum_{j=1}^{N}\mu_j(Q_r(y))}{r^d}=\liminf_{r\to\infty}\inf_{r=|Q_r(y)|}\frac{\sum_{j=1}^{N}\#(\Lambda_j\cap Q_r(y))}{r^d}.$$

□

Corollary 3.7. *If* $\bigcup_{j=1}^{N}\mathscr{E}(g_j,\mu_j)$ *where* $\mu_j=\mu$ *for all* j *forms a frame for* $L^2(\Omega)$ *then* $D^-(\mu)=\frac{|S|}{N\cdot C_3}$. *In particular if* $\mu=\delta_\Lambda$ *then* $D^-(\Lambda)\geq\frac{|\Omega|}{N}$.

Proof.

$$\begin{aligned}
N\cdot D^-(\mu) &= N\cdot\liminf_{r\to\infty}\inf_{r=|Q_r(y)|}\frac{\mu(Q_r(y))}{r^d}\\
&= \liminf_{r\to\infty}\inf_{r=|Q_r(y)|}\frac{N\cdot\mu(Q_r(y))}{r^d}\\
&= D^-(N\mu)\\
&= D^-\left(\sum_{j=1}^{N}\mu_j\right)\\
&\geq\frac{|S|}{C_3}.
\end{aligned}$$

□

One of the limitations of sampling theory when being applied to real-world applications is the requirement on the sampling points being taken arbitrarily, due to physical constraints. Therefore the Beurling density of the sampled points is often below $|\Omega|$.

However in many applications, the signal is known to evolve with respect to time under a certain time evolution operator. Dynamical sampling exploits this fact to improve upon classic sampling. Using an under-sampled set of points and this time evolution operator, one can generate a large enough set for sampling. A practical question to ask now is, if $a^n * f(x) = f_n(x)$, the signal at time $t = n$, what is a lower bound on n so that stable sampling can occur? That is to say given a signal f and the kernel of a time evolution operator a, for what choice of N will $\bigcup_{j=1}^{N} \mathscr{E}(a^j, \delta_X)$ where X is a discrete set be a frame for $L^2(\Omega)$? From Corollary 3.7 we see that if $\bigcup_{j=1}^{N} \mathscr{E}(a^j, \delta_X)$ is a frame for $L^2(\Omega)$ that $D^-(\delta_X) \geq \frac{|\Omega|}{C_3 \cdot N}$. Next by Corollary 3.6 we can further reduce this to $D^-(\delta_X) \geq \frac{|\Omega|}{N}$. Noting that $D^-(\delta_X) = D^-(X)$, we have the following necessary condition $N \geq \frac{|\Omega|}{D^-(X)}$.

Example 3.1. Let $\Omega = [-\frac{1}{2}, \frac{1}{2})$ and $X = m\mathbb{Z}$, and let a be the kernel of a time evolution operator, with $\bigcup_{j=1}^{N} \mathscr{E}(a^j, \delta_X)$ forming a frame for $L^2(\Omega)$. Then because $|\Omega| = 1$, and $D^-(X) = \frac{1}{m}$ we see that $N \geq \frac{1}{\frac{1}{m}} = m$.

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