# PHASE AND CONJUGATE PHASE RETRIEVAL ON PALEY-WIENER SPACES AND $\mathbb{C}^M$

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *PHASE AND CONJUGATE PHASE RETRIEVAL ON PALEY-WIENER SPACES AND* $\mathbb{C}^M$ by Luke Evans and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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Phase retrieval is the recovery of unknown signals from measurements with noisy or lost phase. Recovery from loss of phase occurs in applications such as X-ray crystallography, optics, speech processing, and quantum information theory.

In this thesis, we introduce the concept of conjugate phase retrieval in complex vector spaces and provide examples of real-valued vectors which allow conjugate phase retrieval but not phase retrieval. We completely characterize conjugate phase retrievable vectors in $\mathbb{C}^2$. In Paley-Wiener spaces, we exhibit a connection between sets of uniqueness and unsigned sampling. We prove that a set of uniqueness in $PW^1_{\Omega+\Omega}$ allows unsigned sampling in $PW^2_{\Omega}$, and provide examples suggesting the converse is true. We further show that complex phase retrieval is not possible using real-valued samples for functions in Paley-Wiener spaces.

I certify that the Abstract is a correct representation of the content of this thesis.

[signature] May 23 2017

Chair, Thesis Committee — Date

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# Chapter 1

# Introduction

In many applications where measurements are taken from frames, noisy or corrupted data necessitate reconstruction from possibly incomplete measurements. When full reconstruction of the signal is not possible, one can still try to determine the best possible reconstruction. In particular, reconstruction from loss of phase measurements is known as the **phase retrieval problem**. The classical formulation comes from applications such as X-ray crystallography where a signal must be recovered from the magnitudes of its Fourier coefficients [15]. Phase retrieval also occurs in numerous other applications such as diffraction imaging [7], optics [14], speech processing [5], and quantum information theory [18].

In 2006, Balan, Casazza and Edidin introduced the following formulation for the phase retrieval problem within a complex Hilbert space $\mathcal{H}$ [5]:

**Problem.** Given a set of vectors $\{\varphi_n\}_{n\in I} \subseteq \mathcal{H}$ with index set $I \subseteq \mathbb{N}$, does

$$|\langle x, \varphi_n\rangle| = |\langle y, \varphi_n\rangle|, \text{ for all } n \in I \tag{1.1}$$

imply that $x = e^{i\theta}y$?

In general, recovering $x, y$ fully (i.e, with $x = y$) from only magnitudes of measurements is not a realistic goal since equation (1.1) implies that $x$ has the same magnitude measurement as any $e^{i\theta}y, 0 \leq \theta \leq 2\pi$. Therefore, we require that equation (1.1) implies that $x$ and $y$ are equivalent up to some global phase. In other words, (1.1) should require that $x = e^{i\theta}y$ for some fixed $0 \leq \theta \leq 2\pi$? If equation (1.1) implies that $x$ and $y$ are equivalent up to a global phase, we say that the frame $\{\varphi_n\}_{n\in I}$ is **phase retrievable**. To simplify notation we define an equivalence relation $\sim$ where $x \sim y$ if there exists some $e^{i\theta}$ such that $x = e^{i\theta}y$.

Balan, Casazza, and Edidin [5] showed with algebraic geometry that any generic set of $2M-1$ vectors is phase retrievable in $\mathbb{R}^M$. More specifically, any full spark frame with at least $2M-1$ vectors is phase retrievable.

**Definition 1.1.** We say a frame $\{\varphi_n\}_{n\in I}$ is **full spark** if every subset of $\{\varphi_n\}_{n\in I}$ of size $M$ is linearly independent.

Full spark frames with at least $2M-1$ vectors are of strong interest because they are a *generic* class of frames with the complement property.

**Definition 1.2.** A frame $\{\varphi_n\}_{n=1}^{N}$ in $\mathcal{H}^M$ has the **complement property** if for every index subset $I \subset \{1, \ldots, N\}$ we have $\{\varphi_n\}_{n \in I}$ or $\{\varphi_n\}_{n \in I^c}$ spans $\mathcal{H}^M$. Balan, Casazza and Edidin introduced the complement property as a necessary and sufficient condition for a frame to be phase retrievable in $\mathbb{R}^M$.

However, the complement property is not sufficient for phase retrieval in $\mathbb{C}^M$ and the minimum number of vectors required for phase retrieval is roughly $4M$ with exact number dependent on $M$ [18]. Phase retrieval is much more complicated in complex vector spaces, which one can see from the following comparison: for $x, y \in \mathcal{H}^M$, $x \sim y$ if and only if

$$x = \pm y, \qquad \mathcal{H} \text{ a real hilbert space}$$
$$x = e^{i\theta} y, 0 \leq \theta < 2\pi \qquad \mathcal{H} \text{ a complex Hilbert space.}$$

We further mention that in general, $x \in \mathbb{C}^M$ is not equivalent to its conjugate.

**Example 1.1.** In general, a complex vector is not equivalent to its conjugate modulo a global phase: there is no $\theta$ such that $e^{i\theta}(1+i \quad 1)^T = (1-i \quad 1)^T$.

Thus, measurement vectors that cannot distinguish conjugates will not be complex phase retrievable. For any $\varphi \in \mathbb{R}^M$ $|\langle x, \varphi\rangle|_{\mathbb{C}^M} = |\langle \overline{x}, \varphi\rangle|_{\mathbb{C}^M}$ for all $x \in \mathbb{C}^M$. Thus, frames of all real-valued vectors are never complex phase retrievable.

Hence, no frame of real valued vectors can be complex phase retrievable in a complex vector space. For a frame of real valued vectors $\{\varphi_n\}_{n=1}^N$, $|\langle x, \varphi_n\rangle|_{\mathbb{C}^M} = |\langle \overline{x}, \varphi_n\rangle|_{\mathbb{C}^M}$ for any $x \in \mathbb{C}^M$. Thus, $\{\varphi_n\}_{n=1}^N$ cannot be complex phase retrievable, since otherwise we would conclude $x \sim \overline{x}$ for all $x \in \mathbb{C}^M$. Hence, we propose:

**Definition 1.3.** We say that $\{\varphi_n\}_{n\in I}$ **is conjugate phase retrievable** if $|\langle x, \varphi_n\rangle| = |\langle y, \varphi_n\rangle|$, for all $n \in I$ implies that $x \sim y$ or $x \sim \overline{y}$.

We now show that real-valued frames over $\mathbb{C}^M$ can be conjugate phase retrievable.

**Example 1.2.** The frame $\Phi = \{(1\ 0)^T, (0\ 1)^T, (1\ 1)^T\}$ is conjugate phase retrievable over $\mathbb{C}^2$ (see Chapter 4).

A number of questions arise: what is a necessary and sufficient condition for conjugate phase retrieval on $\mathbb{C}^M$? Though the complement property is an equivalent condition for phase retrievable frames over $\mathbb{R}^M$, we exhibit a connection to conjugate phase retrieval.

**Proposition 1.1.** *Every conjugate phase retrievable frame in $\mathbb{C}^M$ consisting of all real vectors must have the complement property.*

A further question is: what are the frames in $\mathbb{C}^M$ which are conjugate phase retrievable and not phase retrievable? We say that a frame is **strictly conjugate phase retrievable** if it is not complex phase retrievable but is still conjugate phase retrieval. In our results, we prove a strong connection to the set of vectors equivalent to their conjugates modulo a global phase.

**Theorem 1.2.** *Let $\Phi = \{\varphi_n\}_{n=1}^N$ be a conjugate phase retrievable frame in $\mathbb{C}^M$ where $\varphi_n = (\varphi_{1n} \quad \varphi_{2n} \quad \cdots \quad \varphi_{Mn})^T$ for $n \in [N]$. Then, $\Phi$ is strictly conjugate* *phase retrievable if and only if there exists some* $x = (x_1 \ \cdots \ x_M)^T \in \mathbb{C}^M$, *with* $x \not\sim \overline{x}$ *and*

$$\sum_{i<j} \operatorname{Im}(x_i\overline{x_j}) \operatorname{Im}(\varphi_{in}\overline{\varphi_{jn}}) = 0$$

*for each* $n \in [N]$.

We can apply this theorem to study conjugate phase retrieval on $\mathbb{C}^2$. In particular, we prove the following characterization of real-valued strictly conjugate phase retrievable frames on $\mathbb{C}^2$ :

**Theorem 1.3.** *A real-valued frame* $\Phi$ *on* $\mathbb{C}^2$ *is strictly conjugate phase retrievable if and only if* $\Phi$ *has the complement property.*

Our study of conjugate phase retrieval is motivated from unsigned sampling problems in Paley-Wiener spaces. With the Paley-Wiener theorem we may connect phase retrieval in $L^2[-\frac{1}{2}, \frac{1}{2}]$ with sampling theory for entire functions band-limited to $[-\frac{1}{2}, \frac{1}{2}]$. The Paley-Wiener space can be further generalized to any bounded $\Omega \subset \mathbb{R}$.

Let $\Lambda$ be a countable subset of $\mathbb{R}$. We say that $\Lambda$ is a **set of unsigned sampling** for $PW_\Omega$ is the samples $\{|f(\lambda)|\}_{\lambda\in\Lambda}$ uniquely determine $f$ up to a global phase given any $f \in PW_\Omega$.

We generalize Proposition 1 in [2] given by Alaifari, Daubechies, Grohs, and Theorem 2.5 in [1] by Alaifari and Grohs with the following result:

**Theorem 1.4.** *If* $\Lambda \subseteq \mathbb{R}$ *is a set of uniqueness for* $PW^1_{\Omega+\Omega}$, *then* $\Lambda$ *is a set of unsigned sampling on* $PW^2_\Omega$.

We first give an overview of frame theory, then basic definitions and theorems concerning finite-dimensional phase retrieval in Chapter 2. In Chapter 3, we define Paley-Wiener spaces and elementary concepts in sampling theory. Then, we prove Theorem 1.4 and state evidence for a possible converse. In Chapter 4, we define conjugate phase retrieval on $\mathbb{C}^M$ and then state our main result for frames in $\mathbb{C}^2$.

# Chapter 2

# Frames and Phase Retrieval

Frame theory abstracts many signal processing notions such as phase retrieval into a cohesive theory harnessing linear algebra and functional analysis to enrich a wide range of applications. Abstract frame theory resides in Hilbert space, so we will begin with basic Hilbert space notions.

## 2.1 General Frame Theory

Let $\mathcal{H}$ be a vector space over a field $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathcal{H}$ such that $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ forms a Hilbert space. We say that an inner product space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a **Hilbert space** if every Cauchy sequence in $\mathcal{H}$ converges with respect to the norm $||x|| = \sqrt{\langle x, x \rangle}$.

Hilbert spaces arose as a generalization of standard Euclidean space $\mathbb{R}^n$ with inner product given by the standard dot product. In Euclidean space, the dot product of two vectors can be expressed in terms of the angle between two vectors.

One may view an inner product as an abstract "angle" between two vectors in Hilbert space, and hence a Hilbert space can be thought of as an abstract vector space with some notion of angle.

Throughout the thesis, let $\mathcal{H}$ be a Hilbert space over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The set $I$ given in the notation $\{v_n\}_{n\in I}$ for a set of vectors $v_n \in \mathcal{H}$ will always denote a countable index set which may be finite or infinite.

The structure of a Hilbert space comes from inner products. In particular, a vector in $\mathcal{H}$ may be approximated or fully described by a sequence of inner products with some set of vectors in $\mathcal{H}$.

**Definition 2.1.** Given a set of vectors $\{v_n\}_{n\in I}$ in $\mathcal{H}$, we say that $\{v_n\}_{n\in I}$ is **complete** (or **total**) in $\mathcal{H}$ if $\{v_n\}_{n\in I}$ if $\overline{\operatorname{span}\{v_n\}_{n\in I}} = \mathcal{H}$. We say that a set $\{v_n\}_{n\in I}$ in $\mathcal{H}$ is **orthogonal** if $\langle v_i, v_j\rangle = 0$ if $i \neq j$, and **orthonormal** if $\{v_n\}_{n\in I}$ is orthogonal and $||v_n||^2 = \langle v_n, v_n\rangle = 1$ for all $n \in I$.

**Definition 2.2.** Suppose $\mathcal{H}$ is a finite dimensional Hilbert space. We say that a set $\{v_n\}_{n=1}^N$ is a **spanning set** if $\operatorname{span}\{v_n\}_{n=1}^N = \mathcal{H}$. Since every subspace is closed in a finite-dimensional vector space, the spanning set and complete set definitions are equivalent for finite-dimensional vector spaces. Given a spanning set $\{v_n\}_{n=1}^N$ in $\mathcal{H}$, each vector $v \in \mathcal{H}$ may be described as some linear combination of terms in $\{v_n\}_{n=1}^N$. A spanning set that is also linearly independent in $\mathcal{H}$ is called a **basis**. Each $v \in \mathcal{H}$ has a *unique* representation as linear combination of elements in a given basis $\{v_n\}_{n=1}^N$. A set of vectors $\{e_n\}_{n=1}^N$ in $\mathcal{H}$ is an **orthonormal basis** for $\mathcal{H}$ if

$\{e_n\}_{n=1}^{N}$ is a complete and orthonormal set.

**Definition 2.3.** Suppose $\mathcal{H}$ is an infinite dimensional Hilbert space. Then, we say that a sequence $\{v_n\}_{n=1}^{\infty}$ is a **Schauder basis** for $\mathcal{H}$ if for every $v \in \mathcal{H}$ there exists a unique sequence of scalars $\{\alpha_n\}_{n=1}^{\infty}$ where

$$\left|\left|v - \sum_{n=1}^{N} \alpha_n v_n\right|\right| \to 0 \text{ as } n \to \infty.$$

Since we will use Schauder bases and no other basis definitions for infinite-dimensional Hilbert spaces in this thesis, we will further refer to any Schauder basis as a **basis**. Hence, an **orthonormal basis** for $\mathcal{H}$ refers to a orthonormal Schauder basis.

A basis for any given Hilbert space $\mathcal{H}$ admits an expansion of any vector into a linear combination of basis elements, but an orthonormal basis admits an explicit linear combination of basis elements to describe vectors.

**Proposition 2.1.** *Let $\{e_n\}_{n\in I}$ be an orthonormal basis for $\mathcal{H}$. Then, each $v \in \mathcal{H}$ may be uniquely written as $\sum_{n\in I} \langle v, e_n\rangle\, e_n$.*

Given an orthonormal basis $\{e_n\}_{n\in I}$, the sequence of scalars $\{\langle v, e_n\rangle\}_{n\in I}$ reconstructs the norm of $v \in \mathcal{H}$. The reconstruction is known as Parseval's Identity [19]:

**Proposition 2.2** (Parseval's identity, [19]). *Let $\{e_n\}_{n\in I}$ be an orthonormal basis for $\mathcal{H}$. Then, for every $v \in H$,*

$$\sum_{n\in I} |\langle v, e_n\rangle|^2 = ||v||^2.$$

In Parseval's identity we describe the norm of $v$ with scalars originating from an orthonormal basis. However, we may allow reconstruction of $v$ in terms of other spanning sets, and in certain contexts reconstructing $v$ from a spanning set may be preferable to reconstructing $v$ from a basis. The concept of a **frame** for a Hilbert space was originally introduced by Duffin and Schaeffer in their study of non-harmonic Fourier series [13]. The frame notion arises naturally from generalizations of Parseval's identity.

**Definition 2.4.** [10] Let $\{v_n\}_{n\in I}$ be a sequence of vectors in $\mathcal{H}$. The sequence $\{v_n\}_{n\in I}$ is called a **Bessel sequence** for $\mathcal{H}$ if there exists some $B > 0$ such that $\sum_{n\in I} |\langle x, v_n\rangle|^2 \leq B||x||^2$ for all $x \in \mathcal{H}$. The sequence $\{v_n\}_{n\in I}$ is a **frame** for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||x||^2 \leq \sum_{n\in I} |\langle x, v_n\rangle|^2 \leq B||x||^2$$

for all $x \in \mathcal{H}$. We say that $A, B$ are the **lower and upper frame bounds** of $\{v_n\}_{n\in I}$ respectively. A frame with bounds $A = B$ is said to be a **tight frame**. Further, we say a frame $\{v_n\}_{n\in I}$ is a **Parseval frame** if $\{v_n\}_{n\in I}$ has frame bounds $A = B = 1$.

*Remark.* Note that an orthonormal basis is a Parseval frame, since $A = B = 1$ gives Parseval's identity. However, a frame may be Parseval and not an orthonormal basis. Consider the standard orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $\ell^2(\mathbb{N})$. The set of vectors

$\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots\}$ is still a Parseval frame for $\ell^2(\mathbb{N})$ but no longer an orthonormal basis.

**Proposition 2.3.** [10] *A Parseval frame* $\{v_n\}_{n\in I}$ *on* $\mathcal{H}$ *is an orthonormal basis for* $\mathcal{H}$ *if and only if there exists some* $n \in I$ *with* $||v_n|| = 1$.

Since frames and spanning sets are equivalent in finite dimensions, one may think of the frame definition as a finer lens to distinguish classes of spanning sets in finite dimensions. However, in infinite dimensions we cannot interchange these defnitions.

**Proposition 2.4.** [10] *Let* $\mathcal{H}$ *be a Hilbert space.*

1. *Suppose* $\mathcal{H}$ *is finite dimensional. Then, the set* $\{v_n\}_{n\in I}$ *is a frame for* $\mathcal{H}$ *if and only if* $\{v_n\}_{n\in I}$ *is a spanning set for* $\mathcal{H}$.

2. *Suppose* $\mathcal{H}$ *is infinite dimensional. Then, every frame on* $\mathcal{H}$ *is a complete set for* $\mathcal{H}$, *but the converse does not hold.*

**Example 2.1.** Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for an infinite dimensional Hilbert space $\mathcal{H}$. Then the set $\{\frac{e_n}{n}\}_{n=1}^{\infty}$ is complete in $\mathcal{H}$, but not a frame for $\mathcal{H}$. The set $\{\frac{e_n}{n}\}_{n=1}^{\infty}$ is complete, since

$$v = \sum_{n=1}^{\infty} \langle v, e_n \rangle e_n = \sum_{n=1}^{\infty} \langle v, ne_n \rangle \left(\frac{1}{n}\right) e_n$$

for each $v \in \mathcal{H}$.

However, with $v = e_k$, some $k \in \mathbb{Z}$ we have $||v||^2 = 1$ and

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \left\langle v, \frac{e_n}{n} \right\rangle \right|^2 &= \sum_{n=1}^{\infty} \frac{1}{n^2} |\langle v, e_n \rangle|^2 \\ &= \frac{1}{k^2}. \end{aligned}$$

Given, any $A > 0$, there exists $k \in \mathbb{Z}$ such that for $v = e_k$, $\sum\limits_{n=1}^{\infty} \left|\left\langle v, \frac{e_n}{n}\right\rangle\right|^2 \leq A||v||^2$. Thus, the set $\{\frac{e_n}{n}\}_{n=1}^{\infty}$ does not have a lower frame bound.

**Definition 2.5.** Given a frame $\{v_n\}_{n\in I}$ for $\mathcal{H}$ and $x \in \mathcal{H}$, we say that the elements of $\{\langle x, v_n \rangle\}_{n\in I}$ are the **frame coefficients** of $x$ with respect to the frame $\{v_n\}_{n\in I}$.

A frame that is not a basis of $\mathcal{H}$ is not linearly independent and hence gives non-unique representations of each vector of $\mathcal{H}$. Any frame that is not linearly independent is called **over-complete**, or **redundant**. The study of frame theory originally grew as mathematicians began to find over-completeness an asset in many applications, particularly in time-frequency analysis , wavelet analysis and signal processing ( [12]). The reader can refer to [9] or [10] for more in-depth discussions on frame theory.

The analysis and synthesis operators of a set of vectors $\{v_n\}$ specify reconstruction and spanning properties of $\{v_n\}$.

**Definition 2.6.** Given a set of vectors $\Phi = \{v_n\}_{n\in I}$ in $\mathcal{H}$, we define the following operators:

| **Analysis Operator** | **Synthesis Operator** | **Frame Operator** |
|---|---|---|
| $T_\Phi : \mathcal{H} \to \ell^2(M)$ | $T_\Phi^* : \ell^2(M) \to \mathcal{H}$ | $T_\Phi^* T_\Phi : \mathcal{H} \to \mathcal{H}$ |
| $x \mapsto \{\langle x, v_n \rangle\}$ | $\{c_n\} \mapsto \sum c_n v_n$ | $x \mapsto \sum \langle x, v_n \rangle v_n$ |

The analysis operator can be thought of the measurement operator of a frame in that it describes a signal by a sequence of frame coefficients. The synthesis operator is the operator taking measurement values and reconstructing the signal. The frame operator denoted by $S_\Phi = T_\Phi^* T_\Phi$ represents the entire process of breaking down and reconstructing a given signal.

**Theorem 2.5** ([10], Corollary 5.5.3). *The sequence $\Phi$ is a frame for $\mathcal{H}$ if and only if $T_\Phi$ is injective and the range of $T_\Phi$ is closed. If $\mathcal{H}$ is finite-dimensional, then $\Phi$ is a frame if and only if $T_\Phi$ is injective.*

In other words, for finite dimensional $\mathcal{H}$, $\Phi$ is a frame if and only if the frame coefficients $\langle x, v_n \rangle$ uniquely determine the vector $x$.

**Theorem 2.6.** [10] *Given a frame $\Phi = \{\varphi_n\}_{n \in I}$ with frame operator $S$, every vector $x \in \mathcal{H}$ can be written as*

$$x = \sum_{n \in I} \langle x, S^{-1} \varphi_n \rangle \varphi_n .$$

**Definition 2.7.** The sequence of vectors $\{S^{-1}\varphi_n\}_{n\in I}$ is called the **canonical dual frame** of $\Phi$ and is also a frame for $\mathcal{H}$.

Note that the frame reconstruction in Theorem 2.6 is extremely similar to the reconstruction of vectors by an orthonormal basis shown in Proposition 2.1. However, unlike with an orthonormal basis the coefficients $\langle x, S^{-1}\varphi_n\rangle$ are not necessarily unique in that there may exist a different sequence $\{c_n\}$ with $x = \sum c_n\varphi_n$.

**Definition 2.8.** We say a frame $\Phi = \{\varphi_n\}_{n\in I}$ is a **Riesz basis** if for every $x \in \mathcal{H}$ there exists a *unique* sequence of scalars such that $x = \sum_{n\in I} c_n\varphi_n$. The sequence $\{c_n\}_{n\in I}$ can be written in terms of the canonical dual frame of $\Phi$, exactly the sequence $c_n = \langle x, S^{-1}\varphi_n\rangle$ . A frame is called an **exact frame** if it is no longer a frame after the removal of any frame vector.

**Proposition 2.7.** *A set of vectors $\Phi = \{\varphi_n\}_{n\in I}$ is an exact frame for $\mathcal{H}$ if and only if $\Phi$ is a Riesz basis for $\mathcal{H}$.*

For more detailed discussion on Riesz bases the reader may refer to [22] or [17]. We will return to the notion of Riesz bases in Section 3.2.2.

## 2.2 General Phase Retrieval

We may interpret a vector $x \in \mathcal{H}$ as some signal to recover and a frame $\Phi = \{\varphi_n\}_{n\in I} \subseteq \mathcal{H}$ as some collection of measuring devices. The frame coefficients

$\{\langle x, \varphi_n \rangle\}$ are interpreted as physical measurements of the signal $x$. Thus, Proposition 2.5 implies that the frame coefficients of $x$ with respect to $\Phi$ uniquely determine the vector $x$. In X-ray crystallography, diffraction measurements often lose phase information and only retain their magnitudes. The problem of recovering the signal from only magnitudes of measurements is known as the **phase retrieval problem**. The phase retrieval problem can be formulated in terms of frames.

Given a frame $\{\varphi_n\}_{n \in I}$, we ask how well we can approximate or reconstruct $x \in \mathcal{H}$ from the magnitudes of its frame coefficients $\{|\langle x, \varphi_n \rangle|\}_{n \in I}$. We can think of the sequence $\{|\langle x, \varphi_n \rangle|\}_{n \in I}$ as a collection of phaseless measurements on the signal $x$.

*Remark.* Note that perfect reconstruction of the signal $x$ is impossible for non-zero $x$. Let $\lambda \in \mathbb{F}$ with $|\lambda| = 1$, where $\mathbb{F}$ is the field of scalars for $\mathcal{H}$. Given any nonzero $x, y \in \mathcal{H}$ we have

$$|\langle x, y \rangle| = |\lambda| |\langle x, y \rangle| = |\langle \lambda x, y \rangle|.$$

In particular this holds with $y = \varphi_n$ for any $\varphi_n \in \Phi$. Thus, given any $\lambda \in \mathbb{F}$ with $|\lambda| = 1$, $x$ and $\lambda x$ have the same phaseless measurements.

Throughout this thesis, we will interchangeably refer to such a $\lambda \in \mathbb{F}$ with $|\lambda| = 1$ as a **unimodular scalar** or **global phase** and will also use the notation $\lambda \in \mathbb{T}$, where $\mathbb{T} = \{\lambda \in \mathbb{F} \mid |\lambda| = 1\}$. The frame theoretical definition of phase retrieval first developed by Balan, Casazza and Edidin in 2006 [5], considers perfect reconstruction *up to a global phase:*

**Definition 2.9.** [5] Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $\Phi = \{\varphi_n\}_{n\in I}$ be a frame in $\mathcal{H}$. We say that $\Phi$ is real/complex **phase retrievable** if for all $x, y \in \mathcal{H}$,

$$|\langle x, \varphi_n\rangle| = |\langle y, \varphi_n\rangle| \quad \text{for all } n \in I \tag{2.1}$$

implies that $x, y$ are equivalent up to a global phase:

$$\begin{cases} x = e^{i\theta}y & \text{if } \mathbb{F} = \mathbb{C},\ 0 \leq \theta < 2\pi \\ x = \pm y & \text{if } \mathbb{F} = \mathbb{R}. \end{cases}$$

Now, we define a relation $\sim$ on $\mathcal{H}$ where $x \sim y$ if and only if $x = \lambda y$ for some unimodular scalar $\lambda$. With the equivalence relation $\sim$ we consider the set of all equivalence classes with respect to $\sim$ on $\mathcal{H}$, denoted $\mathcal{H}/\sim$ . Analogous to the analysis operator, we define the phaseless analysis operator with respect to a frame $\Phi$.

**Definition 2.10.** Let $\mathcal{H}$ be a Hilbert space of $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $\Phi = \{v_n\}_{n\in I}$ be a frame for $\mathcal{H}$. Then, the **phaseless analysis operator** of $\Phi$ as

$$\mathcal{A}_\Phi : \mathcal{H}/\sim \;\rightarrow \ell^2(I)$$
$$x \mapsto (|\langle x, v_n\rangle|)_{n\in I},$$

Thus, we may further say that a frame $\Phi = \{v_n\}_{n\in I}$ is phase retrievable if and only if the phaseless analysis operator $A_\Phi$ is injective over $\mathcal{H}/\sim$ .

The complement property is a widely used necessary and sufficient condition for phase retrieval. For a set $X$ indexed by $I$, let $X_S$ denote the set $\{x_s \in X \mid s \in S\}$. Futher, let $S^c$ denote the complement of $S$ in $I$.

**Definition 2.11.** Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $\Phi = \{\varphi_n\}_{n \in I}$ be a frame over $\mathcal{H}$. We say that $\Phi$ has the **complement property** if for each subset $S \subseteq I$, one of $\Phi_S$ or $\Phi_{S^c}$ is complete in $\mathcal{H}$.

The following theorem is the fundamental theorem in the frame-theoretic phase retrieval community.

**Theorem 2.8.** [5] *Let $\Phi = \{\varphi_n\}_{n \in I}$ be a frame over a Hilbert space $\mathcal{H}$. If $\Phi$ is phase retrievable, then $\Phi$ has the complement property.*

*Proof.* We proceed by contrapositive. Suppose $\Phi$ does not have the complement property. Then, there exists some $S \subseteq I$ such that $\operatorname{span} \Phi_S \neq \mathcal{H}$ and $\operatorname{span} \Phi_{S^c} \neq \mathcal{H}$. Hence, we can find non-zero $x, y \in \mathcal{H}$ with $\langle x, \varphi_n \rangle = 0$ for all $n \in S$ and $\langle y, \varphi_n \rangle = 0$ for all $n \in S^c$. Note that $x + y \neq 0$ and $x - y \neq 0$, since $x + y = 0$ or $x - y = 0$, would imply that $\langle x, \varphi_n \rangle = 0$ for all $n \in I$, and contradict the fact that $\Phi$ is a frame. We consider the measurements

$$|\langle x + y, \varphi_n \rangle| = \begin{cases} |\langle y, \varphi_n \rangle| & n \in S \\ |\langle x, \varphi_n \rangle| & n \in S^c \end{cases}$$

$$|\langle x-y,\varphi_n\rangle| = \begin{cases} |\langle y,\varphi_n\rangle| & n\in S \\ |\langle x,\varphi_n\rangle| & n\in S^c. \end{cases}$$

Thus, $|\langle x+y,\varphi_n\rangle| = |\langle x-y,\varphi_n\rangle|$ for each $n \in I$. Suppose $x+y = \lambda(x-y)$ for some unimodular scalar $\lambda$. Then, $(1-\lambda)x = -(1+\lambda)y$. Since $x+y, x-y$ are non-zero we can say that $\lambda \neq 1$ and that $\frac{(\lambda-1)}{(\lambda+1)}x = y$. But, $cx = y$ for any constant $c$ implies that $\Phi$ is not a frame. Therefore, we conclude that $x+y \neq \lambda(x-y)$ for any unimodular scalar $\lambda$. Thus, $\Phi$ is not phase retrievable.

□

Since $x \sim y$ implies $x = \pm y$ when considering a real Hilbert space, much stronger results are available for phase retrieval in real Hilbert spaces, especially concerning the complement property.

**Theorem 2.9.** [5] *Let $\mathcal{H}$ be a real Hilbert space and let $\Phi$ be a frame over $\mathcal{H}$. Then, $\Phi$ is phase retrievable if and only if $\Phi$ has the complement property.*

*Proof.* By Theorem 2.8, if $\Phi$ is phase retrievable than $\Phi$ has the complement property. Again, we proceed by contrapositive. Suppose that $\Phi$ is not phase retrievable. Then,there exist some $x, y \in \mathcal{H}$ such that $|\langle x,\varphi_n\rangle| = |\langle y,\varphi_n\rangle|$ for all $n \in I$ but $x \neq \lambda y$ for any unimodular scalar $\lambda$. Since $\mathcal{H}$ is a real Hilbert space, $\lambda = \pm 1$ and $x \neq \pm y$. Let $S = \{n \in I \mid \langle x,\varphi_n\rangle = \langle y,\varphi_n\rangle\}$.Then, $S^c = \{n \in I \mid \langle x,\varphi_n\rangle = -\langle y,\varphi_n\rangle\}$, which follows from the fact that $\mathcal{H}$ is a real Hilbert space. We claim that neither $\Phi_S$ nor $\Phi_{S^c}$ spans $\mathcal{H}$. Consider $x+y, x-y \in \mathcal{H}$. Since $x \neq \pm y$, we have $x+y, x-y \neq 0$.

For $n \in S$, $\langle x - y, \varphi_n \rangle = 0$, and $\langle x + y, \varphi_n \rangle = 0$ for $n \in S^c$. Thus, $\Phi_S$ and $\Phi_{S^c}$ are not complete in $\mathcal{H}$, which implies that $\Phi$ does not have the complement property.

□

### 2.2.1 Finite-Dimensional Phase Retrieval

In finite dimensions we can make more precise estimations on what frames have the complement property. By definition, no basis for $\mathcal{H}$ can have the complement property. Intuitively, any set of vectors with the complement property must have a large number of vectors to have a large number of spanning subsets. We can say more specifically:

**Proposition 2.10.** *Any frame with the complement property over $\mathcal{H} = \mathbb{F}^M$ must have at least $2M - 1$ vectors.*

*Proof.* Suppose $\Phi = \{\varphi_i\}_{i=1}^N$ in $\mathbb{F}^M$ has the complement property. Then, $|I| \geq M$ or $|I^c| = N - |I| \geq M$ for any $I \subseteq [N]$, since otherwise $\Phi_I$ and $\Phi_{I^c}$ are not complete in $\mathbb{F}^M$. The complement property implies that $N - |I| \geq M$ for any $I \subseteq [N]$ with $|I| < M$. In particular, $I = M - 1$ implies that $N \geq 2M - 1$. □

The phase retrievable frames on $\mathbb{R}^M$ are characterized exactly by the complement property. However, the complement property is just a necessary requirement for phase retrieval over $\mathbb{C}^M$.

For use in applications, it is helpful to find some suitably large class of phase retrievable frames. Equivalently, we search for a class of matrices.

*Remark.* Note that we may consider any $M \times N$ matrix $\Phi$ over $\mathbb{F}^M$, as a collection of vectors $\{\varphi_n\}_{n=1}^N$ where $\varphi_n$ is the $n$-th column of $\Phi$. Likewise we can consider any frame $\{\varphi_n\}_{n=1}^N$ on $\mathbb{F}^M$ as the $M \times N$ matrix

$$\begin{bmatrix} | & | & \cdots & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_N \\ | & | & \cdots & | \end{bmatrix}.$$

For the remaining discussion we will interchangeably refer to $\Phi$ as a set of vectors $\{\varphi_n\}_{n=1}^N$ and as an $M \times N$ matrix.

In applications, it is useful to determine if random measurements will have desired measurement properties. To describe the frames more clearly we consider vector spaces of matrices. If $\mathbb{F} = \mathbb{R}$, we consider $\mathbb{F}^{M\times N} = \mathbb{R}^{M\times N}$ endowed with the standard Euclidean topology. If $\mathbb{F} = \mathbb{C}$, we instead consider the space $\mathbb{F}^{M\times N} = \mathbb{R}^{2M\times 2N}$ with the Euclidean topology by considering the real and imaginary parts of each complex entry as separate coordinates.

**Definition 2.12.** Let $X \subseteq \mathbb{F}^{M\times N}$ be the set of $M \times N$ matrices over $\mathbb{F}$ with a certain property called property $(*)$. If $X$ is open and dense in $\mathbb{F}^{M\times N}$ and $\mathbb{F}^{M\times N} \backslash X$ is measure 0, we say that each frame $\Phi \in X$ is called a **generic frame** with property $(*)$.

Ideally, the number of measurements $N$ should be as small as possible. Thus, we wish to find both

1. A minimum $N$ such that any generic frame $\Phi \in \mathbb{F}^{M\times N}$ is phase retrievable.

2. An example of a generic class of phase retrievable frames $X \subseteq \mathbb{F}^{M\times N}$ on $\mathbb{F}^M$.

On $\mathbb{R}^M$, Theorem 2.9 implies that we only need to find a generic class of frames that have the complement property. For $\mathbb{R}^M$, the family of **full spark** frames with at least $2M-1$ vectors is phase retrievable.

**Definition 2.13.** We define the **the spark** of a frame as the size of its smallest linearly independent subset. A frame $\Phi$ in $\mathbb{F}^M$ is **full spark** if spark $\Phi = M+1$, i.e, every subset of $\Phi$ of size $M$ is linearly independent.

**Proposition 2.11.** *A full spark frame in $\mathbb{R}^M$ with at least $2M-1$ vectors has the complement property and hence is phase retrievable.*

**Theorem 2.12.** [4] *The class of full spark frames over $\mathbb{R}^M$ is generic in the set of frames over $\mathbb{R}^M$. Hence, a generic frame with at least $2M-1$ is phase retrievable on $\mathbb{R}^M$, and any frame with less than $2M-1$ vectors is not phase retrievable.*

*Remark.* Theorem 2.12 implies that a frame $\Phi$ over $\mathbb{R}^M$ with entries chosen independently from any probability density function will be full spark with probability 1. In particular, if $\Phi$ has at least $2M-1$ vectors then $\Phi$ is phase retrievable with probability 1. Thus, the complement property completely determines the generic number of vectors required for the real case.

**Example 2.2.** Full spark frames can be constructed quite easily from Vandermonde matrices. An $M \times N$ matrix $\Phi$ with coefficents in $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is said to be a

**Vandermonde** matrix if it has form

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_N \\ \vdots & \vdots & \ddots & \cdots \\ \alpha_1^{M-1} & \alpha_2^{M-1} & \cdots & \alpha_N^{M-1} \end{bmatrix},$$

where $\alpha_1, \ldots, \alpha_N \in \mathbb{F}$.

**Proposition 2.13.** [4] *An $M \times N$ Vandermonde matrix in $\mathbb{R}^M$ or $\mathbb{C}^M$ is full spark if and only if its bases are distinct.*

With Theorem 2.12, we can say that $2M - 1$ is a sharp lower bound for the generic number of vectors needed for phase retrieval in $\mathbb{R}^M$. By sharp we mean that a generic frame $\Phi$ on $\mathbb{R}^M$ with $N$ vectors, $N < 2M - 1$ implies $\Phi$ cannot is not phase retrievable. For a more in depth treatment of full spark frames see [4].

Phase retrieval on complex vector spaces requires substantially more vectors for generic phase retrieval. Recall that for $x, y \in \mathbb{R}^M$, $x \sim y$ if and only if $x = \pm y$, while for $x, y \in \mathbb{C}^M$ $x \sim y$ if and only if $x = e^{i\theta} y$ for some $0 \leq \theta < 2\pi$. For a complex Hilbert space, the complement property is still necessary for phase retrieval but far from sufficient. As it turns out, roughly twice as many vectors will suffice for complex phase retrieval. Since $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, it makes intuitive sense that roughly $2(2M - 1) = 4M - 2$ vectors are required in $\mathbb{C}^M$, although the complex case is actually much more complicated.

**Theorem 2.14.** [11] *A generic frame with at least $4M-4$ vectors is complex phase retrievable on $\mathbb{C}^M$.*

*Remark.* The $4M-4$ bound is not a sharp bound in the complex case: in 2015 it was discovered that there exists a conjugate phase retrievable frame of $11 = 4(4)-5$ vectors for $\mathbb{C}^4$ [21].

### 2.2.2 Infinite-Dimensional Phase Retrieval

Robustness and stability of phase retrieval is extremely important in applications. In particular, the distance between two vectors up to a global phase should be controlled by the distance between the phaseless measurements of the vectors.

Recall that given a frame $\Phi$ for a Hilbert space $\mathcal{H}$ we define the phaseless measurements of $\Phi$ by

$$\mathcal{A}_\Phi : \mathcal{H}/\sim \to \mathbb{R}^{|I|}$$

$$x \mapsto \{|\langle x, \varphi_n\rangle|\}_{n\in I},$$

where $|I|$ denotes the cardinality of the index set $I$ which may be infinite.

**Definition 2.14.** We say that a frame $\Phi = \{\varphi_n\}_{n\in I}$ over $\mathcal{H}$ is **robust in phase retrieval** if there exists some $C > 0$ such that for any $x, y \in \mathcal{H}$ we have

$$\inf_{|\alpha|=1} ||x-\alpha y|| \leq C||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||.$$

Roughly speaking this equation implies that we want to find a lower Lipschitz bound for the phaseless analysis operator with respect to a global phase. In finite dimensions such a Lipschitz bound exists.

**Proposition 2.15.** [8] *Let $\Phi$ be a phase retrievable on $\mathbb{F}^M$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then there exists $C > 0$ such that*

$$\inf_{|\alpha|=1} ||x - \alpha y|| \leq C||\mathcal{A}_\Phi(x) - \mathcal{A}_\Phi(y)||.$$

*for all $x, y \in \mathbb{F}^M$.*

However, no such stability exists for frames in infinite dimensions.

**Theorem 2.16.** [8] *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $\Phi$ be a frame for $\mathcal{H}$ with some $c > 0$ such that $||\varphi|| > c > 0$ for each $\varphi \in \Phi$. Then, for any $\delta > 0$ there exist $x, y \in \mathcal{H}$ such that $\inf_{|\alpha|=1} ||x - \alpha y|| \geq 1$ but $||A_\Phi(x) - A_\Phi(y)|| < \delta$.*

Although phase retrieval is always unstable in infinite dimensions, there are still strong efforts to continue studies in applications of high-dimensional and infinite dimensional phase retrieval. New paradigms for stability are developed in [3] and [2] along with alternative measurements. In the next chapter, we focus on infinite dimensional Hilbert spaces of band-limited functions and analogous phase retrieval concepts for unsigned sampling of real-valued functions.

# Chapter 3

# Phase Retrieval on Paley-Wiener Spaces

We introduce Paley-Wiener spaces and concepts from sampling theory. Then, we define unsigned sampling and prove a theorem closely related to recent results of Alaifari, Daubechies, Grohs, and Thakur in [1] and [2]. Further, we give a conjecture on properties of unsigned sampling sequences corresponding to Riesz bases.

## 3.1 Preliminaries

### 3.1.1 Fourier Frames and $PW_\Omega^p$.

Frame theory began in the study of sequences of exponential functions. The first applications of frames came from a connection between sampling band-limited functions and taking inner products with exponential functions. The same dualities between spaces of band-limited functions and sequences of exponential functions carries over to problems in phase retrieval.

**Definition 3.1.** Given $\Omega \subseteq \mathbb{R}^M$, we define

$$L^p(\Omega) = \left\{ f : \mathbb{R}^M \to \mathbb{C} \middle| \int_\Omega |f(x)|^p dx < \infty \right\}.$$

Note that we define $L^p(\Omega)$ to include complex valued functions on $\mathbb{R}$.

For the following discussion, we consider the Hilbert space $L^2(\Omega)$ with inner product $\langle f, g\rangle := \int_\Omega f(x)\overline{g(x)}dx$. Previously we primarily discussed general orthonormal bases for Hilbert spaces. A prominent example of orthonormal bases for a Hilbert space is the collection of exponential functions $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ considered over $L^2[-\frac{1}{2}, \frac{1}{2}]$. Different texts may instead consider the set$\{e^{\pi inx}\}_{n\in\mathbb{Z}}$ over $L^2[-\pi, \pi]$, or with interval $[0, 2\pi]$.

**Theorem 3.1.** *The family of exponential functions* $\{e^{2\pi i\langle n,x\rangle}\}_{n\in\mathbb{Z}^M}$ *is an orthonormal basis for* $L^2[-\frac{1}{2}, \frac{1}{2}]^M$.

In 1952, Duffin and Schaefer began the study of frames in their work on **Fourier Frames** [13].

**Definition 3.2.** Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathbb{R}^M$. We say the family of exponentials $\{e^{2\pi i\langle\lambda_n,x\rangle}\}_{n=1}^\infty$ is a **Fourier frame** for $L^2[-\frac{1}{2}, \frac{1}{2}]^M$ if there exists constants $A, B > 0$ such that

$$A \int_{[-\frac{1}{2},\frac{1}{2}]^M} |f(x)|^2 dx \leq \sum_{n=1}^\infty \left| \int_{[-\frac{1}{2},\frac{1}{2}]^M} f(x) e^{2\pi i\langle\lambda_n,x\rangle} dx \right|^2 \leq B \int_{[-\frac{1}{2},\frac{1}{2}]^M} |f(x)|^2 dx.$$

With $\langle\cdot,\cdot\rangle$ denoting the $L^2$ inner product, the previous inequality becomes our previous frame definition applied to $L^2[-\frac{1}{2},\frac{1}{2}]^M$ :

$$A||f||^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, e^{2\pi i\langle\lambda_n,x\rangle}\rangle|^2 \leq B||f||^2.$$

**Definition 3.3.** For a function $f \in L^2(\mathbb{R}^M)$, we define the **Fourier transform** of $f$ as

$$\hat{f}(\xi) = \int_{\mathbb{R}^M} f(x)e^{-2\pi i\langle\xi,x\rangle}dx.$$

Given $g \in L^2(\mathbb{R}^M)$ we define the **inverse Fourier transform** of $g$ as

$$\check{g}(x) = \int_{\mathbb{R}^M} g(\xi)e^{2\pi i\langle x,\xi\rangle}d\xi.$$

Any $\hat{f}$ and $\check{g}$ are in $L^2(\mathbb{R}^M)$ by **Plancherel's Theorem:**

**Theorem 3.2** (Plancherel's Theorem). *Let $f \in L^2(\mathbb{R}^M)$. Then, $f$ and its Fourier transform $\hat{f}$ have the same modulus in $L^2(\mathbb{R}^M)$:*

$$||f||^2_{L^2} = \int_{\mathbb{R}^M} |f(t)|^2dt = \int_{\mathbb{R}^M} |\hat{f}(\xi)|^2d\xi = ||\hat{f}||^2_{L^2}$$

**Definition 3.4.** Let $\Omega$ be a bounded, measurable subset of $\mathbb{R}^M$. Then, let

$$PW^p(\Omega) = \{f \in L^p(\mathbb{R}^M) | \operatorname{supp}\hat{f} \in \Omega\}.$$

The following result combined with Plancherel's theorem gives the fundamental duality between $PW^2(\Omega)$ and $L^2(\Omega)$.

**Theorem 3.3** (Fourier Inversion,[22]). *Let $f \in PW^2(\Omega)$, and define $g(x) = \hat{f}(x)$. Then, $g \in L^2(\Omega)$ and $f = \check{g}$. Further, for $g \in L^2(\Omega)$ we have $\check{g} \in PW^2(\Omega)$.*

*Remark.* Plancherel's theorem and Theorem 3.3 give the **Paley-Wiener Isometry**, the duality between $PW^2(\Omega)$ and $L^2(\Omega)$. For any $f \in PW^2(\Omega)$, $||f|| = ||\hat{f}||$ and $\hat{f} \in L^2(\Omega)$. From the other direction, given any $g \in L^2(\Omega)$, there exists $f \in PW^2(\Omega)$ such that $g = \hat{f}$ and $||f|| = ||g||$.

### 3.1.2 Sampling on $PW^p_\Omega$

A hallmark of sampling theory is the duality between the set of samples $\Lambda \subseteq \mathbb{R}^M$ and the set of exponential functions $\{e_\lambda\}_{\lambda \in \Lambda}$. For the remainder of the chapter we let $e_\lambda$ denote the function $e^{2\pi i \langle \lambda, x \rangle}$, given some $\lambda \in \mathbb{R}^M$. For $b > 0$, $PW^p_b$ denotes $PW^p[-\frac{b}{2}, \frac{b}{2}]$ and $L^p_b$ denotes $L^2[-\frac{b}{2}, \frac{b}{2}]$. Also for some set $\Omega \subset \mathbb{R}$ denote $PW^p(\Omega)$ and $L^p(\Omega)$ by $PW^p_\Omega$, $L^p_\Omega$ respectively.

**Definition 3.5.** Let $\Lambda$ be a countable subset of $\mathbb{R}$ and let $\Omega$ be a bounded subset of $\mathbb{R}$. We say that $\Lambda$ is a **set of uniqueness** for $PW^p_\Omega$ if given $f \in PW^p_\Omega$, $f(\lambda) = 0$ for all $\lambda \in \Lambda$ implies that $f$ is identically 0 on $\mathbb{R}$.

**Definition 3.6.** We say that a countable set $\Lambda \subseteq \mathbb{R}^M$ is **exact** in $PW^p_\Omega$ if $\Lambda$ is a set of uniqueness for $PW^p_\Omega$ but that $\Lambda \backslash \{\lambda\}$ given any $\lambda \in \Lambda$ is no longer a set of uniqueness. Analogously, we define a sequence $\{e_\lambda\}_{\lambda\in\Lambda}$ in $L^p_\Omega$ as **exact** if the sequence is no longer complete after the removal of any term.

**Proposition 3.4.** [20] *The set $\Lambda$ is a set of uniqueness in $PW^2_\Omega$ if and only if the functions $\{e_\lambda\}_{\lambda\in\Lambda}$ are complete in the space $L^2_\Omega$.*

*Proof.* We prove the result by contrapositive. Suppose that $\Lambda$ is not a set of uniqueness for $PW^2_\Omega$. Then, there exists some $f \in PW^2_\Omega$ not identically 0 such that $f(\Lambda) = \{0\}$. Hence, by Theorem 3.3 we have

$$\langle \hat{f}, e_{-\lambda}\rangle = \int_\Omega \hat{f}(\xi)e^{2\pi i\langle\lambda,\xi\rangle}d\xi = f(\lambda) = 0$$

for all $\lambda \in \Lambda$. However, $\langle \hat{f}, e_{-\lambda}\rangle = 0$ if and only if $\langle \overline{\hat{f}}, e_\lambda\rangle = \overline{\langle \hat{f}, e_{-\lambda}\rangle} = 0$, so we can say that $\langle \overline{\hat{f}}, e_\lambda\rangle = 0$ for each $\lambda \in \Lambda$. Since $f$ is non-trivial, we can say that $\overline{\hat{f}}$ is non trivial. Hence, $\{e_\lambda\}_{\lambda\in\Lambda}$ is not complete in $L^2_\Omega$.

Now, suppose that the functions $\{e_\lambda\}_{\lambda\in\Lambda}$ are not complete in $PW^2_\Omega$. Then, there exists some $F \in PW^2_\Omega$ not identically 0 such that $\langle F, e_\lambda\rangle = 0$ for all $\lambda \in \Lambda$. We know that there exist some non-trivial $f \in PW^2_\Omega$ such that $\hat{f} = \overline{F}$. Therefore, $f(\lambda) = \langle \overline{F}, e_{-\lambda}\rangle = \langle F, e_\lambda\rangle = 0$ for all $\lambda \in \Lambda$. Hence, $\Lambda$ is not a set of uniqueness for $PW^2_\Omega$. □

**Definition 3.7.** Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a countable subset of $\mathbb{R}$. Also, let $\Omega$ be a bounded subset of $\mathbb{R}^M$. We say that $\Lambda$ is a **set of sampling** for $PW^2_\Omega$ if there exist

$A, B > 0$ such that

$$A||f||_{L^2}^2 \leq \sum_{n=1}^{\infty} |f(\lambda_n)|^2 \leq B||f||_{L^2}^2$$

for all $f \in PW_\Omega^2$.

**Proposition 3.5.** [20] *The set $\Lambda$ is a set of sampling for $PW_\Omega^2$ if and only if the set $\{e_\lambda\}_{\lambda\in\Lambda}$ is a frame for $L_\Omega^2$.*

*Proof.* Suppose $\Lambda$ is a set of sampling for $PW_\Omega^2$. Then there exist $A, B > 0$ such that $A||f||^2 \leq \sum_{\lambda\in\Lambda} |f(\lambda)|^2 \leq B||f||^2$ for each $f \in PW_\Omega^2$. Since $f(\lambda) = \langle \hat{f}, e_{-\lambda}\rangle$ and $||f|| = ||\hat{f}||$ by Plancherel's Theorem, we may write $A||\hat{f}||^2 \leq \sum_{\lambda\in\Lambda} |\langle \hat{f}, e_{-\lambda}\rangle|^2 \leq B||\hat{f}||^2$. Hence, we can say that $A||\overline{\hat{f}}||^2 \leq \sum_{\lambda\in\Lambda} |\langle \overline{\hat{f}}, e_\lambda\rangle|^2 \leq B||\overline{\hat{f}}||^2$, where $\overline{\hat{f}} \in L_\Omega^2$.

Theorem 3.3 implies that for every $g \in L_\Omega^2$ there exist $f \in PW_\Omega^2$ with $\hat{f} = g$. Thus, we can say that

$$A||g||^2 \leq \sum_{\lambda\in\Lambda} |\langle g, e_\lambda\rangle|^2 \leq B||g||^2$$

for all $g \in L_\Omega^2$. Therefore, $\{e_\lambda\}_{\lambda\in\Lambda}$ is a frame for $L_\Omega^2$.

If we instead assume $\{e_\lambda\}_{\lambda\in\Lambda}$ is a frame for $L_\Omega^2$ we can use identical reasoning to find that $\Lambda$ is a set of sampling for $PW_\Omega^2$. □

*Remark.* We know that given a frame $\{\varphi_n\}_{n\in I}$ over a Hilbert space $\mathcal{H}$, each $v \in \mathcal{H}$ is uniquely determined by the frame coefficients $\{\langle v, \varphi_n\rangle\}_{n\in I}$ (see page 13). By the above theorem, the samples $\{f(\lambda)\}_{\lambda\in\Lambda}$ uniquely determine $f$ for all $f \in PW_\Omega^2$ exactly when the inner products $\{\langle g, e_{\lambda_n}\rangle\}_{n\in I}$ uniquely determine all $g \in L_\Omega^2$.

We have defined $L^p(\mathbb{R})$ to include complex valued functions on $\mathbb{R}$, and hence $PW_\Omega^p$ includes complex valued functions defined on $\mathbb{R}$. By the Paley-Wiener theorem, every $f \in PW_a^2$ can be considered as an function defined on all of $\mathbb{C}$ and entire.

**Theorem 3.6** (Paley-Wiener). [20] *Every function $f \in PW_a^2$ can be extended to an entire function $F$ with*

$$|F(x+iy)| \le Ce^{\frac{a|y|}{2}}, \qquad C > 0 \textit{ depending on } f, \quad \textit{for all } x, y \in \mathbb{R}.$$

The Shannon sampling theorem is classic theorem of sampling theory and motivates the results in the following section.

**Theorem 3.7** (Shannon Sampling Theorem). *Each $f \in PW_1^2$ has the unique expansion*

$$f(z) = \sum_{n\in\mathbb{Z}} f(n)\frac{\sin(\pi(z-n))}{\pi(z-n)}.$$

*Remark.* We use the notation $\text{sinc}(z) := \frac{\sin(z)}{z}$.

## 3.2 Unsigned Sampling on $PW_\Omega^p$

From the Shannon Sampling theorem, any function $f \in PW_1^2$ is uniquely determined by its samples on $\mathbb{Z}$, $\{f(n)\}_{n\in\mathbb{Z}}$. In this section, we generalize to unsigned samples

$\{|f(\lambda)|\}_{\lambda\in\Lambda}$ on real valued $f \in PW^2_\Omega$.

**Definition 3.8.** Let $\Omega$ be a bounded subset of $\mathbb{R}$, and let $\Lambda$ be a countable subset of $\mathbb{R}$. We denote the set of real-valued functions in $PW^p_\Omega$ by ${}_{\mathbb{R}}PW^p_\Omega$. We say that $\Lambda$ is an **unsigned sampling set**(abbreviated USS) for ${}_{\mathbb{R}}PW^p_\Omega$ if for any $f, g \in {}_{\mathbb{R}}PW^p_\Omega$, $|f(\lambda)| = |g(\lambda)|$ for all $\lambda \in \Lambda$ implies that $f = \pm g$.

**Definition 3.9.** We can define the complement property for sampling analogously to the previous definition for Hilbert spaces. We say a countable set $\Lambda \subseteq \mathbb{R}$ has the **complement property** for ${}_{\mathbb{R}}PW^p_\Omega$ if for any subset $X$ of $\Lambda$ we have $X$ or $\Lambda \backslash X$ is a set of uniqueness for ${}_{\mathbb{R}}PW^p_\Omega$.

**Proposition 3.8.** *A bounded set $\Lambda \subseteq \mathbb{R}$ has the complement property in ${}_{\mathbb{R}}PW^2_\Omega$ if and only if the set of exponentials $\{e_\lambda\}_{\lambda\in\Lambda}$ has the complement property in $L^2_\Omega$.*

*Proof.* This follows immediately from Proposition 3.4. □

Though the complement property characterizes phase retrievable frames for real Hilbert spaces, analogous characterizations for unsigned sampling sequences are more difficult to devise.

### 3.2.1 Sets of Uniqueness and Unsigned Sampling

Before introducing the main theorem on sets of uniqueness, we give a basic fact on Paley-Wiener spaces.

**Proposition 3.9.** *For $\Omega \subseteq \mathbb{R}$, $\Omega$ bounded, $PW_\Omega^1 \subseteq PW_\Omega^2$.*

To prove Proposition 3.9 we refer to the following result of Young in the Corollary on page 87 of [22] :

**Lemma 3.10.** [22] *Any function in $PW_\Omega^p$ must be bounded.*

*Proof of Lemma 3.10.* Now, suppose $f \in PW_\Omega^1$. Then, $f \in L^1(\mathbb{R})$ and by Lemma 3.10 there exists $C > 0$ such that $|f(x)| \leq C$ for all $x \in \mathbb{R}$. Hence,

$$\begin{aligned}\int_\Omega |f(x)|^2 dx &= \int_\Omega |f(x)||f(x)|dx \\ &\leq C \int_\Omega |f(x)|dx \\ &< \infty.\end{aligned}$$

Thus, $f \in PW_\Omega^2$. □

In the following theorem, we generalize Proposition 1 in [2] and Theorem 2.5 in [1] to general $\Omega$.

**Theorem 3.11.** *Let $\Omega \subseteq \mathbb{R}$ and let $\Lambda \subseteq \mathbb{R}$. If $\Lambda$ is a set of uniqueness for $PW_{\Omega+\Omega}^1$, then $\Lambda$ is a USS for ${}_{\mathbb{R}}PW_\Omega^2$.*

*Proof.* We proceed by contradiction. Suppose that $\Lambda$ is a set of uniqueness for $PW_{\Omega+\Omega}^1$, but that $\Lambda$ is not an unsigned sampling set in $PW_\Omega^2$. Then, there exist $f, g \in {}_{\mathbb{R}}PW_\Omega^2$ such that $|f(\lambda)| = |g(\lambda)|$ for all $\lambda \in \Lambda$ but $f \neq \pm g$. Let $A = \{x \in \Lambda \mid$

$f(x) = g(x)\}$ and let $B = \Lambda \backslash A = \{x \in \Lambda \mid f(x) = -g(x)\}$. With $u(x) = f(x) - g(x)$ and $v(x) = f(x) + h(x)$, we see that $u$ is identically 0 on $A$ and $v$ is identically 0 on $B$. However, $f \neq \pm g$ implies that neither $u$ nor $v$ are trivial functions.

Let $w(x) = u(x)v(x)$. Then, $w$ is identically 0 on $\Lambda$. Further, $u, v \in L^2(\mathbb{R}^M)$ implies that $\operatorname{supp} \hat{w} \subseteq (\Omega + \Omega)$ and that $w \in L^1(\mathbb{R}^M)$ by Holder's inequality. Hence, $w \in PW^1_{\Omega+\Omega}$. Since $\Lambda$ is a set of uniqueness on $PW^1_{\Omega+\Omega}$, we can say that $w$ is identically 0. But, the Paley-Wiener theorem implies that $u$ and $v$ are both entire functions, so it follows that either $u$ is identically 0 or $v$ is identically 0. We assumed originally that $u$ and $v$ are non-trivial functions, which gives a contradiction. □

From Proposition 3.9 we can apply Theorem 3.11 to sets of uniqueness in $PW^2_{\Omega+\Omega}$.

**Corollary 3.12.** *Let $\Omega, \Lambda$ be defined as in Theorem 3.11. If $\Lambda$ is a set of uniqueness for $PW^2_{\Omega+\Omega}$, then $\Lambda$ is a USS for ${}_{\mathbb{R}}PW^2_{\Omega}$.*

*Question* 1. Is the converse to Theorem 3.11 true? Given a USS $\Lambda$ on ${}_{\mathbb{R}}PW^2_{\Omega}$, does it follow that $\Lambda$ is a set of uniqueness for $PW^1_{\Omega+\Omega}$?

We first consider the case with $\Omega = [a, b]$ :

**Example 3.1.** Let $[a, b] \subseteq \mathbb{R}$ with $b - a < \frac{1}{2}$. Then $\mathbb{N}$ is a USS for ${}_{\mathbb{R}}PW^2[a, b]$. This follows from a result given by Young:

**Proposition 3.13** ([22], page 96). *The set $\{e^{2\pi inx}\}_{n=1}^{\infty}$ is complete in $L^p$ over every interval of length less than 1.*

By construction, $|[a,b]+[a,b]| = |[2a,2b]| < 1$. From the proposition in [22],the set $\{e^{2\pi inx}\}_{n\in\mathbb{N}}$ is a set of uniqueness for $L^2([a,b]+[a,b]) = L^2_{[2a,2b]}$. We can apply Proposition 3.11 to conclude that $\mathbb{N}$ is a USS for ${}_{\mathbb{R}}PW^2[a,b]$

**Example 3.2.** The set $\mathbb{Z}$ is a USS for ${}_{\mathbb{R}}PW^2[a,b]$ given any $[a,b]$ with $|b-a| \leq \frac{1}{2}$. We can apply Proposition 3.11 from the fact that $2(b-a) \leq 1$ and that $\mathbb{Z}$ is then a set of uniqueness for $PW^2([a,b]+[a,b])$.

**Proposition 3.14.** [22] *The sequence* $\{e^{2\pi in}\}_{n\in\mathbb{Z}}$ *is a Riesz basis for* $L^p_1$.

Recall that a Riesz basis is a frame which is no longer a frame after the removal of one element. Proposition 3.14 immediately implies that $\mathbb{Z}$ is exact in $PW^2_1$.

**Example 3.3.** By Proposition 3.14, the set $\mathbb{Z}\backslash\{0\}$ is not a set of uniqueness in $PW^2_1$. However, $\mathbb{Z}\backslash\{0\}$ is a set of uniqueness in $PW^1_1$. which implies by Theorem 3.11 that $\mathbb{Z}\backslash\{0\}$ is a USS for ${}_{\mathbb{R}}PW^2_{\frac{1}{2}}$.

Suppose to the contrary. Then, there exists some non-zero $f \in PW^1_1$ such that $f(n) = 0$ for all $n \in \mathbb{Z}\backslash\{0\}$. Proposition 3.9 implies that $f \in PW^1_1$. Thus, by the Shannon sampling theorem we have

$$h(x) = \sum_{n\in\mathbb{Z}} h(n)\,\mathrm{sinc}\,\pi(x-n).$$

Therefore, $h(x) = \mathrm{sinc}\,\pi(x)$. However, it is well known that $\mathrm{sinc}\,\pi(x) \notin L^1(\mathbb{R})$. Hence, we have a contradiction, so we conclude that $\mathbb{Z}\backslash\{0\}$ is a USS for ${}_{\mathbb{R}}PW^1_1$.

**Example 3.4.** In the case of $\mathbb{Z}\backslash\{0\}$ on $\Omega = [-\frac{1}{4}, \frac{1}{4}]$, removal of another element removes the USS property: the set $\mathbb{Z}\backslash\{0, 1\}$ is not a USS for $_{\mathbb{R}}PW^2_{\frac{1}{2}}$. To show this, we give a translation lemma:

**Lemma 3.15.** *The set $2\mathbb{Z}+1$ is exact in $PW^2_{\frac{1}{2}}$.*

*Proof.* For any $n \in \mathbb{Z}$, $f \in PW^2_{\frac{1}{2}}$ we can write

$$f(2n+1) = \int_{-\frac{1}{4}}^{\frac{1}{4}} \hat{f}(x)e^{2\pi i(2n+1)x}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\hat{f}(x)e^{2\pi ix})e^{2\pi i(2n)}dx.$$

Now, define the function $G(x) = \hat{f}(x)e^{2\pi ix}$. Then, we have $\langle G, e_{-2n}\rangle = 0$ for each $n \in \mathbb{Z}$, which implies that $\langle \overline{G}, e_{2n}\rangle = 0$ for all $n \in \mathbb{Z}$. Since $\{e_{2n}\}_{n\in\mathbb{Z}}$ is complete in $L^p_{\frac{1}{2}}$, we can say that $G$ is identically 0. However, $G(x) = \hat{f}e^{2\pi ix}$, where $e^{2\pi ix}$ is non-zero, so we conclude that $\hat{f}$ is identically 0 and that $f$ is identically 0. Hence, $2\mathbb{Z}+1$ is a set of uniqueness for $PW^2_{\frac{1}{2}}$.

Suppose that $f \in PW^2_{\frac{1}{2}}$ with $f(2m+1) \neq 0$ for some $m \in \mathbb{Z}$ but $f(2n+1) = 0$ for all $n \in \mathbb{Z}$ with $n \neq m$. By similar reasoning as before, we consider the function $G(x) = \hat{f}e^{2\pi ix}$. Then, $\overline{G}$ is orthogonal to all $n \in \mathbb{Z}\backslash\{m\}$ with $\langle \overline{G}, e_{2m}\rangle \neq 0$. Since $\{e_{2n}\}_{n\in\mathbb{Z}}$ is exact in $L^p_{\frac{1}{2}}$, $G$ must be non-zero. Since $e^{2\pi ix}$ is non-zero, we can say that $\hat{f}$ is not identically 0 and that $f$ is not identically zero. Thus, $2\mathbb{Z}+1$ is exact in $PW^2_{\frac{1}{2}}$.

□

We claim that $\mathbb{Z}\backslash\{0, 1\}$ does not have the complement property in $_{\mathbb{R}}PW^2_{\frac{1}{2}}$. The set $\mathbb{Z}\backslash\{0,1\}$ can be decomposed as $\mathbb{Z}\{0,1\} = 2\mathbb{Z}\backslash\{0\} \oplus (2\mathbb{Z}+1)\backslash\{1\}$. Since $2\mathbb{Z}$ is exact in $PW^2_{\frac{1}{2}}$, the set $2\mathbb{Z}\backslash\{0\}$ is not a set of uniqueness for $PW^2_{\frac{1}{2}}$. By the lemma, $(2\mathbb{Z}+1)\backslash\{1\}$ is not a set of uniqueness for $PW^2_{\frac{1}{2}}$. Therefore, $\mathbb{Z}\backslash\{0,1\}$ is not a USS for ${}_{\mathbb{R}}PW^2_{\frac{1}{2}}$.

*Remark.* The set $\mathbb{Z}\backslash\{0\}$ is an interesting example: $\{e^{\pi inx}\}_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2_1$, and the sequence $Z\backslash\{0\}$ is still a USS in ${}_{\mathbb{R}}PW^2_{\frac{1}{2}}$ with the removal of one element. But in this case the removal of a second element removes the unsigned sampling property.

**Example 3.5.** For any $a \in \mathbb{Z}$, the set $\mathbb{Z}\backslash\{a\}$ is a USS for ${}_{\mathbb{R}}PW^2_{\frac{1}{2}}$, but the set $\mathbb{Z}\backslash\{a,b\}$ is not a USS for ${}_{\mathbb{R}}PW^2_{\frac{1}{2}}$ given $b \neq a$.

To show this, we can use the argument in Example 3.3, since $\text{sinc}(\pi(x-k)) \notin L^1(\mathbb{R})$ for any fixed $k \in \mathbb{R}$. Further,

$$\mathbb{Z}\backslash\{a,b\} = 2\mathbb{Z}\backslash a \oplus (2\mathbb{Z}+1)\backslash\{b\},$$

and the same reasoning follows as with Example 3.4.

### 3.2.2 Lagrange-Type Interpolation Problem

Examples 3.4 and 3.4 involve an exact sequence which is a set of unsigned sampling after removal of one element but no longer after the removal of a second element. Further, the set $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ forms a Riesz Basis for $L^2_1$. A central theorem concerning

Riesz bases for $L^2_1$ is Kadec's $\frac{1}{4}$-Theorem([22], p. 36.)

**Theorem 3.16** (Kadec's $\frac{1}{4}$-Theorem, [22]). *If $\{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of reals where*

$$|\lambda_n - n| \leq L < \frac{1}{4} \quad n = 0, \pm 1, \pm 2, \ldots,$$

*then $\{e^{2\pi i\lambda_n x}\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $L^2_1$.*

Our first question to motivate further study is: given any $\Lambda$ corresponding to a Riesz basis for $L^2_{\Omega+\Omega}$, does the removal of one and then two elements follows the removal pattern in Examples 3.3 and 3.4?

**Definition 3.10.** We say that $\Lambda \subseteq \mathbb{R}$ is an **exact set of sampling** for $PW^p_\Omega$ if $\Lambda$ is a set of sampling which fails to be a set of sampling after the removal of one element.

**Proposition 3.17.** *The set $\Lambda$ is an exact set of sampling for $PW^2_\Omega$ if and only if the set $\{e_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis for $L^2_\Omega$.*

*Proof.* This follows from Proposition 3.5 and that a frame is a Riesz basis if and only if it ceases to be a frame after the removal of one element. □

*Question* 2. Suppose $\Lambda$ is an exact set of sampling in $PW^2_1$. Then, given $\lambda \in \Lambda$, can we say that $\Lambda\backslash\{\lambda\}$, is a USS for $PW^2_{\frac{1}{2}}$?

*Remark.* A possible approach would be exactly the approach in Example 3.4. Suppose $\Omega = [\frac{-1}{4}, \frac{1}{4}]$, with $\Lambda$ a countable subset of $\mathbb{R}$ and an exact sequence in $PW^2[-\frac{1}{2}, \frac{1}{2}]$.

Suppose that $\Lambda$ is not a USS after removal of $\lambda$. Then, use an analogous sampling theorem applicable to $\Lambda$ to find a non-integrable function $h \in {}_{\mathbb{R}}PW_1^2$ and draw out a contradiction.

We provide the Paley-Wiener-Levinson theorem as a possible route to solving the conjecture.

**Theorem 3.18.** [16] *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$ with $\sup_{n\in\mathbb{Z}} |\lambda_n - n| < \frac{1}{4}$. Further, let*

$$G(z) = (z - \lambda_0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right)\left(1 - \frac{z}{\lambda_{-n}}\right).$$

*Then, $G(z)$ is an entire function with set of zeroes $\Lambda$, and any $f \in PW_1^2$ can written as*

$$f(z) = \sum_{n\in\mathbb{Z}} f(\lambda_n) \frac{G(z)}{G'(\lambda_n)(z - \lambda_n)}.$$

We refer to the sequence $\sum_{n\in\mathbb{Z}} f(\lambda_n) \frac{G(z)}{G'(\lambda_n)(z-\lambda_n)}$ as a **Lagrange-Type Interpolation Series**.

*Remark.* Consider $\Lambda$ as defined in the theorem. By Kadec's $\frac{1}{4}$-Theorem[22], the set $\{e_\lambda\}_{\lambda\in\Lambda}$ in the theorem must be a **Riesz Basis** for $L_1^2$. Hence, $\Lambda$ must be an exact set of sampling for $PW_1^2$.

**Example 3.6.** The Shannon sampling theorem is exactly the Paley-Wiener-Levinson theorem whenever $\Lambda = \mathbb{Z}$. We know that for $f \in PW_1^2$, $f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}$.

Now, let

$$G(z) = \frac{\sin(\pi z)}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Then, equivalently $f(z) = \sum_{n=-\infty}^{\infty} f(n) \frac{G(z)}{G'(z)(z-n)}$. This follows from that fact that $\sin(z) = \cos(\pi n)\sin(z - \pi n)$ for $n \in \mathbb{N}$, $z \in \mathbb{C}$.

In Example 3.3, we used the non-integrability of the functions $\frac{G(z)}{G'(n)(z-n} =$ $\text{sinc}(\pi(z-n)$ to give a contradiction. We conjecture that non-integrability holds for more general $G(z)$ functions and hence for more general sampling sequences $\Lambda$ :

*Question* 3. Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be an exact set of sampling for $PW_1^2$, with

$$G(z) = (z - \lambda_0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\lambda_n}\right) \left(1 - \frac{z}{\lambda_{-n}}\right) \quad \text{for } z \in \mathbb{C}.$$

Is the function $S_n(z) = f(\lambda_n) \frac{G(z)}{G'(\lambda_n)(z - \lambda_n)}$. non-integrable for any $n \in \mathbb{Z}$?

## 3.3 Conjugate Phase Retrieval

Previously, we have only considered real-valued functions $f, g$ and recovering up to $f = \pm g$. However functions in $PW_a^2$ are in general complex-valued. Any $f \in PW_a^2$ can be extended to the complex plane through the Paley-Wiener theorem. A natural extension of our definition of unsigned sampling sets to complex functions could be the following: $\Lambda$ is a set of unsigned sampling if $|f(\lambda)| = |g(\lambda)|$ for all $\lambda \in \Lambda$ implies that $f = ge^{i\theta}$ for some $0 \le \theta < 2\pi$. We claim that unsigned sampling as defined above is impossible in $PW_1^2$.

**Proposition 3.19.** *Let $f, g \in PW_1^2$ such that $f_{\mathbb{R}}, g_{\mathbb{R}}$ are complex-valued. Then, there does not exist a set $\Lambda \subseteq \mathbb{R}$ such that $|f(\lambda)| = |g(\lambda)|$ for each $\lambda \in \Lambda$ implies $f = ge^{i\theta}$ for some $0 \le \theta < 2\pi$.*

*Proof.* Suppose to the contrary that there exist some $\Lambda \subseteq \mathbb{R}$ that is a USS as defined above. Let $f \in PW_1^2$. Then, by the Paley-Wiener theorem we may consider $f$ as a function extended to the complex plane.

Then, the function $g$ defined by $g(z) = \overline{f(\overline{z})}$ is an entire function with $g(x) = f(x)$ for all $x \in \mathbb{R}$. Note also that $f$ and $g$ will agree point-wise in absolute value for the unsigned sampling set $\Lambda$. Further, for each $x$ such that $f(x) \neq 0$, we must then have $\dfrac{f(x)}{g(x)} = e^{i\theta}$. For a fixed $x$, let $f(x) = |f(x)|e^{i\theta_x}$ for some $0 \le \theta_x < 2\pi$. Then, $\dfrac{f(x)}{\overline{f(x)}} = e^{i\theta}$ implies that $e^{2i\theta_x} = e^{i\theta}$ and that $e^{i\theta_x} = \pm e^{\frac{i\theta}{2}}$. Hence

$$f(x) = \pm|f(x)|e^{\frac{i\theta}{2}}.$$

We can say that each $f \in PW_1^2$ can be written as $f = he^{i\theta'}$ for some real-valued function $h$ and some $0 \le \theta' < 2\pi$.

However, it is not true that every function in $PW_1$ is of the form $h(x)e^{i\theta'}$ for a real-valued $h(x)$ and constant $\theta'$. Take the function $f(z) = \operatorname{sinc}(\pi z) + i\operatorname{sinc}(\pi(z-1))$ in $PW_1$. A quick calculation shows that $f(0) = 1$ but that $f(1) = i$. Thus, $f \neq he^{i\theta'}$

for a real-valued function $g$ and some constant $\theta'$. Thus, our complex-valued unsigned sampling cannot work as described on Paley-Wiener spaces. $\square$

This leads to the question of how we could relax our definition of complex unsigned sampling so that we do not require $f = \overline{f}e^{i\theta}$ for $f \in PW_1$. We propose relaxing our definition above to the following:

**Definition 3.11.** Let $\Lambda$ be a countable subset of $\mathbb{R}$. We say $\Lambda$ is a **set of conjugate unsigned sampling** for $PW_b$ if for any $f, g \in PW_b$ $|f(\lambda)| = |g(\lambda)|$ for all $\lambda \in \Lambda$ implies that $f_{\mathbb{R}} = e^{i\theta} g_{\mathbb{R}}$ or $f_{\mathbb{R}} = e^{i\theta}\overline{g}_{\mathbb{R}}$ for some $0 \leq \theta \leq 2\pi$.

Much of our previous results for real unsigned sampling arise from analogies or direct connections with frame-theoretic definitions of phase retrieval. Motivated by the Paley-Wiener spaces example, we return to finite-dimensional vector spaces to investigate the analogous phase retrieval definition of conjugate unsigned sampling.

# Chapter 4

# Conjugate Phase Retrieval on $\mathbb{C}^M$

## 4.1 Introduction

The questions raised by conjugate unsigned sampling on $PW$ can be analogously formulated for vector spaces. We recall that the complement property fully characterizes real phase retrieval, but that the complement property does not guarantee complex phase retrieval. Further, the minimum number of generic measurements for real phase retrieval in $\mathbb{R}^M$ is $2M-1$ but approximately $4M$ for $\mathbb{C}^M$ [?, Wolf] This disparity as well as the less tractable nature of the complex case motivate a relaxed definition of phase retrieval in complex vector spaces.

One motivation for conjugate phase retrieval is the existence of highly redundant frames in $\mathbb{C}^M$ that allow real phase retrieval but not complex phase retrieval. Consider any real frame $\Phi$ on $\mathbb{C}^M$. One may reason intuitively that the lack of complex structure in $\Phi$ would prevent distinguishing of complex structure under phaseless measurements.

**Lemma 4.1.** *Let* $\hat{x}, \hat{\varphi} \in \mathbb{C}^M$ *with* $\hat{\varphi}$ *a real vector in the sense that all entries of* $\varphi$ *are in* $\mathbb{R}$*. Then,* $|\langle \hat{x}, \hat{\varphi} \rangle|_{\mathbb{C}^M} = |\langle \overline{\hat{x}}, \hat{\varphi} \rangle|_{\mathbb{C}^m}$*.*

*Proof.* Since $\varphi = \overline{\varphi}$ and $\overline{\langle x, \varphi \rangle} = \langle \varphi, x \rangle$ we calculate :

$$\begin{aligned} |\langle x, \varphi \rangle|^2 &= \langle x, \varphi \rangle \overline{\langle x, \varphi \rangle} \\ &= \langle x, \varphi \rangle \langle \varphi, x \rangle \\ &= \langle x, \overline{\varphi} \rangle \langle \overline{\varphi}, x \rangle \\ &= \langle \varphi, \overline{x} \rangle \langle \overline{x}, \varphi \rangle \\ &= |\langle \overline{x}, \varphi \rangle|^2. \end{aligned}$$

Hence, $|\langle x, \varphi \rangle|^2 = |\langle \overline{x}, \varphi \rangle|^2$.

□

So, for a real vector $\varphi \in \mathbb{C}^M$ the phaseless measurement $x \mapsto |\langle x, \varphi \rangle|$ ignores conjugation. It follows quickly that phaseless measurements from a real frame distinguish too little among input vectors to allow complex phase retrieval.

**Proposition 4.2.** *Let* $\Phi = \{\varphi_i\}_{i=1}^M$ *be a frame for* $\mathbb{C}^M$ *with* $\Phi \subseteq \mathbb{R}^M$*. Then,* $\Phi$ *does not allow phase retrieval on* $\mathbb{C}^M$*.*

*Proof.* By Lemma 4.1, for any $\hat{x} \in \mathbb{C}^M$, we have that $A(\hat{x}) = A(\overline{\hat{x}})$. Consider the vector $\hat{x} = (i\ 1\ 1\ \cdots\ 1)^T \in \mathbb{C}^M$. With $\overline{\hat{x}} = (-1\ 1\ 1\ \cdots\ )^T$, we can write $\overline{\hat{x}} = (\lambda_1 i\ \lambda_2\ \lambda_2\ \cdots\ \lambda_2)^T$ with $\lambda_1 = -1, \lambda_2 = 1$. Since $\lambda_1 \neq \lambda_2$, we can say that there is no $\lambda \in \mathbb{C}$, $|\lambda| = 1$ with $\hat{x} = \lambda\overline{\hat{x}}$. Since $A(x) = A(\overline{x})$, we conclude $A$ is not injective with respect to the frame $\Phi$. □

However, real vectors may satisfy a relaxed notion of complex phase retrieval which we define below.

**Definition 4.1.** Let $\Phi = \{\varphi_n\}_{n=1}^N$ be a frame in $\mathbb{C}^M$. We say that $\Phi$ is **conjugate phase retrievable** if for all $x, y \in \mathbb{C}^M$,

$$|\langle x, \varphi_n\rangle| = |\langle y, \varphi_n\rangle| \text{ for } n = 1, \ldots M$$

implies that $x = \lambda y$ for some $\lambda \in \mathbb{T}$ or $x = \lambda' \overline{y}$ for some $\lambda' \in \mathbb{T}$.

*Remark.* Note that complex phase retrieval implies conjugate phase retrieval. Let $x \sim y$ denote that $x = \lambda y$ for some $\lambda \in \mathbb{T}$. Then our new definition says that if $x, y$ have the same phaseless measurements with repsect to a conjugate phase retrievable $\Phi$, we must have $x \sim y$ or $x \sim \overline{y}$.

**Example 4.1.** In [6], Bandeira, Cahill, Mixon, and Nelson give the frame

$$\Phi = \left\{(1\ 0)^T, (0\ 1)^T, (1\ 1)^T\right\}$$

as an example of a frame with the complement property but is still not complex phase retrievable for $\mathbb{C}^2$. This follows from Proposition 4.2, since $\Phi$ has all real vectors. However, we now show that $\Phi$ does allow conjugate phase retrieval.

*Proof.* Let $x, y \in \mathbb{C}^2$ with $x = (x_1\, x_2)^T$ and $y = (y_1\, y_2)^T$. Suppose $\{|\langle x, \varphi_i\rangle|^2\}_{i=1}^3 = \{|\langle y, \varphi_i\rangle|^2\}_{i=1}^3$. Then, the following equations hold:

$$|x_1|^2 = |y_1|^2 \tag{4.1}$$

$$|x_2|^2 = |y_2|^2 \tag{4.2}$$

$$|x_1 + x_2|^2 = |y_1 + y_2|^2. \tag{4.3}$$

Using equations (4.1),(4.2), and (4.3) we have

$$\begin{aligned} |x_1 + x_2|^2 &= |y_1 + y_2|^2 \\ |x_1|^2 + 2\operatorname{Re}(x_1\overline{x_2}) + |x_2|^2 &= |y_1|^2 + 2\operatorname{Re}(y_1\overline{y_2}) + |y_2|^2 \\ \operatorname{Re}(x_1\overline{x_2}) &= \operatorname{Re}(y_1\overline{y_2}) \qquad \text{(by (4.1), (4.2))}. \end{aligned}$$

Since $|x_1\overline{x_2}| = |y_1\overline{y_2}|$ by (4.1) and (4.2), we have that $\operatorname{Re}(x_1\overline{x_2})^2 + \operatorname{Im}(x_1\overline{x_2})^2 = \operatorname{Re}(y_1\overline{y_2})^2 + \operatorname{Im}(y_1\overline{y_2})^2$, and with $\operatorname{Re}(x_1\overline{x_2}) = \operatorname{Re}(y_1\overline{y_2})$ we can say that $\operatorname{Im}(x_1\overline{x_2}) = \pm\operatorname{Im}(y_1\overline{y_2})$. Equivalently, $x_1\overline{x_2} = \overline{y_1}y_2$ or $y_1\overline{y_2}$.

Suppose $x_1\overline{x_2} = y_1\overline{y_2}$. Since $|x_1| = |y_1|$ and $|x_2| = |y_2|$, there exist $\lambda_1, \lambda_2 \in \mathbb{C}$

with $1 = |\lambda_1| = |\lambda_2|$ such that $x_1 = \lambda_1 y_1$ and $x_2 = \lambda_2 y_2$. Thus,

$$x_1\overline{x_2} = (\lambda_1 y_1)(\overline{\lambda_2 y_2}) = \lambda_1\overline{\lambda_2}y_1\overline{y_2}.$$

Since $x_1\overline{x_2} = y_1\overline{y_2}$, we can say that $\lambda_1\overline{\lambda_2}y_1\overline{y_2} = y_1\overline{y_2}$. If $y_1 = 0$ or $y_2 = 0$, $x \sim y$ follows trivially. If $y_1$ and $y_2$ are non-zero it follows that $\lambda_1\overline{\lambda_2} = 1$ and that $\lambda_1 = \lambda_2$. Therefore, $x_1 = \lambda_1 y_1$ and $x_2 = \lambda_1 y_2$, which implies that $x \sim y$.

Suppose $x_1\overline{x_2} = \overline{y_1}y_2$. Then, by the same reasoning $x_1 = \lambda_1'\overline{y_1}$ and $x_2 = \lambda'\overline{y_2}$. We can follow the same proof to find that $x \sim \overline{y}$. Therefore, $\{|\langle x, \varphi_i\rangle|^2\}_{i=1}^3 = \{|\langle y, \varphi_i\rangle|^2\}_{i=1}^3$ implies $x \sim y$ or $x \sim \overline{y}$. Hence, $\Phi$ is conjugate phase retrievable.

□

## 4.2 General Results on $\mathbb{C}^M$

In this section, we will develop some general theorems of conjugate phase retrieval on $\mathbb{C}^M$. We begin with some simple facts.

### 4.2.1 Phased Real Vectors

For many of the following results we use the following definition of equivalance up to phase. Let $\hat{x} = (x_1 \ \cdots \ x_M)^T, \hat{y} = (y_1 \ \cdots \ y_M)^T \in \mathbb{C}^M$. Let $\lambda_i$ be the complex scalar such that $y_i = \lambda_i x_i$ for $i \in [M]$, where $[M] = \{1, 2, \ldots, M\}$. Then, we have that $\hat{x} = \lambda\hat{y}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ if and only if

$$|\lambda_k| = 1 \text{ for all } k \in [M] \text{ and } \lambda_1 = \lambda_2 = \cdots = \lambda_M. \tag{4.4}$$

Our results focus on real frames within complex vector spaces. However, we can generalize all of our results to a certain class of complex vectors.

**Lemma 4.3.** *Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Then, for any vectors $x, y, \varphi \in \mathbb{C}^M$ we have*

$$|\langle x, \varphi\rangle|^2 = |\langle y, \varphi\rangle|^2 \textit{ if and only if } |\langle x, \lambda\varphi\rangle|^2 = |\langle y, \lambda\varphi\rangle|^2.$$

*Proof.* Suppose $x, y \in \mathbb{C}^M$ with $|\langle x, \varphi\rangle|^2 = |\langle y, \varphi\rangle|^2$. Then, for any unimodular $\lambda$ we have

$$|\langle x, \lambda\varphi\rangle|^2 = |\lambda|^2|\langle x, \varphi\rangle|^2 = |\lambda|^2|\langle y, \varphi\rangle|^2 = |\langle y, \lambda\varphi\rangle|^2.$$

The converse is immediate from the same equations as above. □

**Proposition 4.4.** *Let $x, \varphi \in \mathbb{C}^M$ with $\varphi = \lambda v$ where $v \in \mathbb{R}^M$ and $\lambda \in \mathbb{T}$. Then,*

$$|\langle x, \varphi\rangle| = |\langle \overline{x}, \varphi\rangle|.$$

*Proof.* The proof is immediate from Lemma 4.1 and Lemma 4.4. □

By Proposition 4.4, we can interchangeably refer to conjugate phase retrievability of a given real-valued frame $\{v_n\}_{n=1}^N$ and any frame of form $\{\lambda_n v_n\}_{n=1}^N$ given unimodular scalars $\{\lambda_n\}_{n=1}^N$.

**Proposition 4.5.** *Suppose $\Phi = \{\varphi_n\}_{n=1}^{M}$ is a frame in $\mathbb{C}^M$ with $\varphi_n = \lambda_n v_n$ for some $\lambda_n \in \mathbb{T}$ and $v_n \in \mathbb{R}^M$ given any $n \in [N]$. Then, $\Phi$ is conjugate phase retrievable on $\mathbb{C}^M$ if and only if for every choice of unimodular scalars $\lambda_1, \ldots, \lambda_M$ the frame $\{\lambda_n \varphi_n\}_{n=1}^{N}$ is conjugate phase retrievable on $\mathbb{C}^M$.*

*Proof.* This follows directly from Lemma 4.3. □

Complex phase retrieval fails using real vectors because there always exist $x$ and $\overline{x}$ that are not equivalent up to phase. A natural question that arises is: what if a vector $x$ and its conjugate $\overline{x}$ have the same phaseless measurements?

**Definition 4.2.** If $y$ is equivalent to its conjugate $\overline{y}$ up to a global phase, we say that $y$ is a **phased real vector**. Let

$$W = \{\lambda \hat{v} \mid \lambda \in \mathbb{T}, \hat{v} \in \mathbb{R}^M\}$$

denote the set of all phased real vectors in $\mathbb{C}^M$.

*Remark.* Proposition 4.5 implies that for phaseless measurements we can refer to any frame $\Phi \subseteq W$ as a frame of real vectors without loss of generality.

**Proposition 4.6.** *A vector $y \in \mathbb{C}^M$ is equivalent to its conjugate $\overline{y}$ up to a global phase if and only if $y \in W$.*

*Proof.* Suppose $y \sim \overline{y}$. Then, $y_m = e^{i\theta}\overline{y_m}$ for $n \in [M]$ where $0 \leq \theta < 2\pi$. Writing $y_m = |y_m|e^{i\theta_m}$ it follows that $e^{i\theta_m} = e^{i\theta - \theta_m}$. Thus,that $2\theta_m = \theta + 2\pi i k$ for some

$k \in \mathbb{N}$, which implies that $\theta_m = \frac{\theta}{2} + \pi i k$. Hence, $e^{i\theta_m} = e^{i\frac{\theta}{2}}$ or $-e^{i\frac{\theta}{2}}$. Therefore, $y_m = \pm e^{i\frac{\theta}{2}}$ for each $m \in [M]$ with sign depending on $m$. Thus, $y = \lambda v$ with $\lambda = e^{i\frac{\theta}{2}}$ and $v = (\pm|y_1| \ \pm |y_2| \ \cdots \ \pm |y_m)^T$. □

*Remark.* Note that Proposition 4.6 implies that no frame $\Phi \subseteq W$ is complex phase retrievable.

### 4.2.2 Strict Conjugate Phase Retrieval

**Definition 4.3.** We say a frame is **strictly conjugate phase retrievable** if the frame is conjugate phase retrievable but not complex phase retrieval.

Strict conjugate phase retrieval relates directly back to phased real vectors. The following property characterizes those frames which strictly allow conjugate phase retrieval:

**Proposition 4.7.** *Suppose that $\Phi = \{\varphi_n\}_{i=1}^N$ is a frame over $\mathbb{C}^M$ that is conjugate phase retrievable. Then, $\Phi$ is strictly conjugate phase retrievable if and only if there exists some $y \in \mathbb{C}^m$ with $y \notin W$ but $|\langle y, \varphi_n\rangle|^2 = |\langle \overline{y}, \varphi_n\rangle|^2$ for all $n \in [N]$.*

*Proof.* Suppose that $\Phi$ is strictly conjugate phase retrievable. Then, there exist $x, y \in \mathbb{C}^m$ such that $|\langle x, \varphi_n\rangle|^2 = |\langle y, \varphi_n\rangle|^2$ for all $n \in [N]$, with $x \not\sim y$ but $x \sim \overline{y}$. Since $\sim$ is transitive , $y \sim \overline{y}$ would imply that $x \sim y$, a contradiction. Hence, $y \not\sim \overline{y}$ and we can then say $y \notin W$.

With $x \sim \overline{y}$ we can say $x = \lambda \overline{y}$ for some unimodular scalar $\lambda$, which gives

$$|\langle y, \varphi_n \rangle|^2 = |\langle \lambda x, \varphi_n \rangle|^2 = |\langle \lambda \overline{y}, \varphi_n \rangle|^2 = |\langle \overline{y}, \varphi_n \rangle|^2.$$

Thus, $|\langle y, \varphi_n \rangle|^2 = |\langle \overline{y}, \varphi_n \rangle|^2$ for all $n \in [N]$ where $y \notin W$.

Conversely, suppose that there exists some $y \in \mathbb{C}^M$ with $y \notin W$ and $|\langle y, \varphi_n \rangle|^2 = |\langle \overline{y}, \varphi_n \rangle|^2$ for all $n \in [N]$. Since $y \notin W$ implies $y \not\sim \overline{y}$ it follows that $\Phi$ is not complex phase retrievable and is only strictly conjugate phase retrievable by the original assumption. □

The following theorem is a characterization of strictly conjugate phase retrievable frames.

**Theorem 4.8.** *Let $\Phi = \{\varphi_n\}_{n=1}^N$ be a conjugate phase retrievable frame in $\mathbb{C}^M$ where $\varphi_n = (\varphi_{1n} \quad \varphi_{2n} \quad \cdots \quad \varphi_{Mn})^T$ for $n \in [N]$. Then, $\Phi$ is strictly conjugate phase retrievable if and only if there exists some $x = (x_1 \quad \cdots \quad x_M)^T \in \mathbb{C}^M$, with $x \notin W$ and*

$$\sum_{i<j} \operatorname{Im}(x_i \overline{x_j}) \operatorname{Im}(\overline{\varphi_{in}} \varphi_{jn}) = 0 \tag{4.5}$$

*for each $n \in [N]$.*

To prove Theorem 4.8 we provide a lemma that gives conditions on vectors $x, \varphi$ where $\varphi$ gives identical phaseless measurements to $x$ and $\overline{x}$.

**Lemma 4.9.** *For* $x = (x_1 \ \cdots \ x_M)^T, \varphi = (\varphi_1 \ \cdots \ \varphi_M)^T \in \mathbb{C}^M$,

$$|\langle x, \varphi\rangle|^2 = |\langle \overline{x}, \varphi\rangle|^2 \text{ if and only if } \sum_{i<j} \operatorname{Im}(x_i\overline{x_j}) \operatorname{Im}(\overline{\varphi_i}\varphi_j) = 0.$$

*Proof of Lemma 4.9.* Let $x = (x_1 \ \cdots \ x_M)^T, \varphi = (\varphi_1 \ \cdots \ \varphi_M)^T \in \mathbb{C}^M$. Expanding using the definition of the conjugate, we may write

$$\begin{aligned}
|\langle x, \varphi\rangle|^2 &= \sum_{i=1}^M x_i\overline{\varphi_i} \sum_{j=1}^M \overline{x_j}\varphi_j \\
&= \sum_{i,j=1}^M x_i\overline{\varphi_i}\overline{x_j}\varphi_j \\
&= \sum_{k=1}^M |x_k\varphi_k|^2 + \sum_{i,j=1, i\neq j}^M x_i\overline{\varphi_i}\overline{x_j}\varphi_j .
\end{aligned}$$

Thus,

$$\begin{aligned}
|\langle x, \varphi\rangle|^2 - |\langle \overline{x}, \varphi\rangle|^2 &= \sum_{i,j=1,\, i\neq j}^M x_i\overline{\varphi_i}\overline{x_j}\varphi_j - \overline{x_i}\overline{\varphi_i}x_j\varphi_j \\
&= \sum_{i,j=1,\, i\neq j}^M \overline{\varphi_i}\varphi_j(x_i\overline{x_j} - \overline{x_i}x_j) \\
&= \sum_{i,j=1,\, i\neq j}^M \overline{\varphi_i}\varphi_j(2i \operatorname{Im}(x_i\overline{x_j}).
\end{aligned}$$

Now, for any fixed $i, j$, we observe that if we swap indices for $i, j$ we have the equality

$\overline{\varphi_i}\varphi_j(2i\,\mathrm{Im}(x_i\overline{x_j})) = \overline{\overline{\varphi_j}\varphi_i(2i\,\mathrm{Im}(x_j\overline{x_i})}$. Therefore, we can split our sum into a sum over indices with $i < j$ and a sum over indices with $j < i$ to find:

$$\begin{aligned}
\sum_{i,j=1,\,i\neq j}^{M} \overline{\varphi_i}\varphi_j 2i\,\mathrm{Im}(x_i\overline{x_j}) &= \sum_{i<j} \overline{\varphi_i}\varphi_j 2i\,\mathrm{Im}(x_i\overline{x_j}) + \sum_{j<i} \overline{\varphi_i}\varphi_j 2i\,\mathrm{Im}(x_i\overline{x_j}) \\
&= \sum_{i<j} \left[\overline{\varphi_i}\varphi_j 2i\,\mathrm{Im}(x_i\overline{x_j}) + \overline{\overline{\varphi_i}\varphi_j 2i\,\mathrm{Im}(x_i\overline{x_j})}\right] \\
&= \sum_{i<j} 4\,\mathrm{Re}(i(\overline{\varphi_i}\varphi_j\,\mathrm{Im}(x_i\overline{x_j}))) \\
&= \sum_{i<j} -4\,\mathrm{Im}(\overline{\varphi_i}\varphi_j\,\mathrm{Im}(x_i\overline{x_j})) \\
&= \sum_{i<j} -4\,\mathrm{Im}(\overline{\varphi_i}\varphi_j)\,\mathrm{Im}(x_i\overline{x_j}). \\
&= \sum_{i<j} -4\,\mathrm{Im}(x_i\overline{x_j})\,\mathrm{Im}(\overline{\varphi_i}\varphi_j).
\end{aligned}$$

Therefore, $|\langle x, \varphi\rangle|^2 = |\langle \overline{x}, \varphi\rangle|^2$ if and only if $\sum_{i<j} \mathrm{Im}(x_i\overline{x_j})\,\mathrm{Im}(\overline{\varphi_i}\varphi_j) = 0$. □

*Proof of Theorem 4.8.* Suppose $\Phi$ is strictly conjugate phase retrievable. By Proposition 4.7 there exists some $x \in \mathbb{C}^M$ with $x \not\sim \overline{x}$ and $|\langle x, \varphi_n\rangle|^2 = |\langle \overline{x}, \varphi_n\rangle|^2$ for $n \in N$. Use of Lemma 4.9 with $x$ and $\varphi_n$ for each $n \in [N]$ completes this direction of the proof.

Suppose there exists a vector $x \notin W$ and that

$$\sum_{i<j} \mathrm{Im}(x_i\overline{x_j})\,\mathrm{Im}(\overline{\varphi_{in}}\varphi_{jn}) = 0 \text{ for each } n \in [N].$$

Then, Lemma 4.9 implies that $|\langle x, \varphi_n\rangle|^2 = |\langle \overline{x}, \varphi_n\rangle|^2$ for each $n \in \mathbb{N}$, which gives that $\Phi$ is not complex phase retrievable. □

Note that given any conjugate phase retrievable $\Phi \subseteq W$, equation (4.5) holds for any $\varphi \in \Phi$ and $x \in \mathbb{C}^M$. Hence, Theorem 4.8 implies $\Phi$ is strictly conjugate phase retrievable. In the next section we show that in $\mathbb{C}^2$, every strictly conjugate phase retrievable frame is a frame in $W$.

### 4.2.3 The Complement Property in $\mathbb{R}^M$ and $\mathbb{C}^M$

Since we deal with frames of real-valued vectors in $\mathbb{C}^M$, we need to derive spanning properties of real vectors in $\mathbb{C}^M$. The relationship between the real span of a real frame in $\mathbb{R}^m$ and the complex span of the same frame in $\mathbb{C}^M$ is crucial for our later results.

**Lemma 4.10.** *A collection of real-valued vectors $\{\varphi_n\}_{n=1}^N$ in $\mathbb{C}^M$ has*

$$\text{span}_{\mathbb{C}}\{\varphi_n\}_{n=1}^N = \mathbb{C}^M \textit{ if and only if } \text{span}_{\mathbb{R}}\{\varphi_n\}_{n=1}^N = \mathbb{R}^m.$$

*Proof.* Let $\{\varphi_n\}_{n=1}^N$ be a collection of real-valued vectors in $\mathbb{C}^M$. We write

$$\text{span}_{\mathbb{C}}\{\varphi_n\}_{n=1}^N = \left\{\sum_{n=1}^N z_n\varphi_n \mid z_n \in \mathbb{C}\right\} = \left\{\sum_{n=1}^N (a_n + ib_n)\varphi_n \mid a_n, b_n \in \mathbb{R}\right\}.$$

After distributing $(a_n + ib_n)\varphi_n$ in the above set, we receive

$$\text{span}_{\mathbb{C}}\{\varphi_n\}_{n=1}^N = \text{span}_{\mathbb{R}}\{\varphi_n\}_{n=1}^N \oplus \text{span}_{\mathbb{R}}\{i\varphi_n\}_{n=1}^N.$$

Suppose that $\{\varphi_n\}_{n=1}^N$ spans $\mathbb{C}^M$. Since $\mathbb{C}^M$ is the direct sum of $\mathbb{R}^m$ and $i\mathbb{R}^M$, we conclude that $\text{span}_{\mathbb{R}}\{\varphi_n\}_{n=1}^N = \mathbb{R}^M$. Conversely, if $\text{span}_{\mathbb{R}}\{\varphi_n\}_{n=1}^N = \mathbb{R}^M$, then $\text{span}_{\mathbb{R}}\{i\varphi_n\}_{n=1}^N = i\mathbb{R}^M$ and we can say that $\text{span}_{\mathbb{C}}\{\varphi_n\}_{n=1}^N = \mathbb{C}^M$. □

**Proposition 4.11.** *Every conjugate phase retrievable frame in $\mathbb{C}^M$ consisting of all real-valued vectors has the complement property in $\mathbb{C}^M$.*

*Proof.* Let $\Phi = \{\varphi_n\}_{n=1}^N$ be a frame of real-valued vectors in $\mathbb{C}^M$ allowing conjugate phase retrieval. For any $x, y \in \mathbb{R}^M$, $|\langle x, \varphi_n\rangle|^2 = |\langle y, \varphi_n\rangle|^2$ for all $n \in [N]$ implies $x \sim y$ or $x \sim \overline{y}$. Since $y$ is real we conclude that $x \sim y$. Hence, $\Phi$ is real phase retrievable on $\mathbb{R}^M$, and must have the complement property in $\mathbb{R}^M$. Since the complement property is defined by spanning properties, Lemma 4.10 implies that $\Phi$ must have the complement property in $\mathbb{C}^M$. □

### 4.2.4 A Characterization By Coordinates

For convenience in notation in later proofs, we introduce the notation

$$x \equiv y \iff x \sim y \text{ or } x \sim \overline{y},$$

which means that $x \equiv y \iff x = \lambda y$ for some unimodular scalar $\lambda$, or $x = \lambda' \overline{y}$ for some unimodular scalar $\lambda'$.

The following theorem gives a coordinate-wise characterization for $x \equiv y$ that we utilize in the next section.

**Theorem 4.12.** *For* $x = (x_1 \ \cdots \ x_M)^T, y = (y_1 \ \cdots \ y_M)^T \in \mathbb{C}^M$,

$$x \equiv y \textit{ if and only if } \mathrm{Re}(x_i \overline{x_j}) = \mathrm{Re}(y_i \overline{y_j}) \quad \textit{for all } i, j \in [M].$$

To prove Theorem 4.12, we first introduce the following lemma:

**Lemma 4.13.** *For* $x = (x_1 \ \cdots \ x_M)^T, y = (y_1 \ \ldots \ y_M)^T \in \mathbb{C}^M$, $\mathrm{Re}(x_i \overline{x_j}) = \mathrm{Re}(y_i \overline{y_j})$ *for all* $i, j \in [M]$ *if and only if*

1. $x_i \overline{x_j} = y_i \overline{y_j}$ *for all* $i, j \in [M]$ *or*

2. $x_i \overline{x_j} = \overline{y_i} y_j$ *for all* $i, j \in [M]$.

*Proof of Lemma 4.13.* Suppose that $x = (x_1 \ \cdots \ x_M)^T$ and $y = (y_1 \ \cdots \ y_M)^T$ are vectors in $\mathbb{C}^M$ where $\mathrm{Re}(x_i \overline{x_j}) = \mathrm{Re}(y_i \overline{y_j})$ for each $i, j \in [M]$. We first claim that given $i, j$, $x_i \overline{x_j} = y_i \overline{y_j}$ or $x_i \overline{x_j} = \overline{y_i} y_j$.

We first note that $|x_i|^2 = \mathrm{Re}(x_i \overline{x_i}) = \mathrm{Re}(y_i \overline{y_i}) = |y_i|^2$ for $i, j \in [M]$ and we can say that $|x_k| = |y_k|$ for all $k \in [M]$. Thus, $|x_i x_j|^2 = |y_i y_j|^2$ given $i, j \in [M]$. With

$\mathrm{Re}(x_i\overline{x_j}) = \mathrm{Re}(y_i\overline{y_j})$ we find

$$\mathrm{Re}(x_i\overline{x_j})^2 + \mathrm{Im}(x_i\overline{x_j})^2 = \mathrm{Re}(y_i\overline{y_j})^2 + \mathrm{Im}(y_i\overline{y_j})^2$$
$$\mathrm{Im}(x_i\overline{x_j}) = \pm\,\mathrm{Im}(y_i\overline{y_j})$$

given any $i, j \in [M]$. Hence, given $i, j \in [M]$ we have $\mathrm{Re}(x_i\overline{x_j}) = \mathrm{Re}(y_i\overline{y_j})$ and $\mathrm{Im}(x_i\overline{x_j}) = \pm\,\mathrm{Im}(y_i\overline{y_j})$. Therefore, $x_i\overline{x_j} = y_i\overline{y_j}$ or $x_i\overline{x_j} = \overline{y_i\overline{y_j}} = \overline{y_i}y_j$ given $i, j \in [M]$.

Now we prove by induction on $M$ that

1. $x_i\overline{x_j} = y_i\overline{y_j}$ for all $i, j \in [M]$ or

2. $x_i\overline{x_j} = \overline{y_i}y_j$ for all $i, j \in [M]$.

First, let $M = 2$ and let $x = (x_1\ x_2)^T, y = (y_1\ y_2)^T$ be vectors in $\mathbb{C}^2$ such that $\mathrm{Re}(x_i\overline{x_j}) = \mathrm{Re}(y_i\overline{y_j})$ for any $i, j \in \{1, 2\}$. From the beginning of the proof it follows that $|x_1|^2 = |y_1|^2, |x_2|^2 = |y_2|^2$ and that $x_1\overline{x_2} = y_1\overline{y_2}$ or $x_1\overline{x_2} = \overline{y_1}y_2$. Trivially we have $x_1\overline{x_1} = y_1\overline{y_1} = \overline{y_1}y_1$ and likewise $x_2\overline{x_2} = y_2\overline{y_2} = \overline{y_2}y_2$. Therefore, $x_i\overline{x_j} = y_i\overline{y_j}$ for all $i, j \in [2]$ or $x_i\overline{x_j} = \overline{y_i}y_j$ for all $i, j \in \{1, 2\}$.

Let $M \in \mathbb{N}$ with $M \geq 2$ and suppose that the claim holds for dimension $M - 1$. Let $x = (x_1\ \cdots\ x_M)^T$ and $y = (y_1\ \cdots\ y_M)^T$ be vectors in $\mathbb{C}^M$ where $\mathrm{Re}(x_i\overline{x_j}) = \mathrm{Re}(y_i\overline{y_j})$ for each $i, j \in [M]$. Consder the vectors $(x_1\ \cdots\ x_{M-1})^T$ and $(y_1\ \cdots\ y_{M-1})^T$ in $\mathbb{C}^{M-1}$. Suppose $x_1\overline{x_2} = y_1\overline{y_2}$. By the inductive hypothesis, $x_i\overline{x_j} = y_i\overline{y_j}$ for all $i, j \in [M-1]$. Similarly, considering the vectors $(x_2\ \cdots\ x_M)^T$ and $(y_2\ \cdots\ y_M)^T$ in $\mathbb{C}^{M-1}$ we conclude by the induction hypothesis that $x_i\overline{x_j} = y_i\overline{y_j}$ for all $i, j \in \{2, \ldots, M\}$.

Hence, combining the conditions on $(x_1 \; \cdots \; x_{M-1})^T$ and $(x_2 \cdots \; x_M)^T$ we have $x_i\overline{x_j} = y_i\overline{y_j}$ for all $i, j \in [M]$. Similarly, if we assume instead that $x_1\overline{x_2} = \overline{y_1}y_2$ we conclude that $x_i\overline{x_j} = \overline{y_i}y_j$ for all $i, j \in [M]$. $\square$

We now proceed to the proof of Theorem 4.12.

*Proof of Theorem 4.12.* Suppose $x, y \in \mathbb{C}^M$ where $x \equiv y$.Then, suppose first that $x \sim y$. We may write $x_k = \lambda y_k$ for all $k \in [M]$, where $\lambda \in \mathbb{T}$. Then, $x_i\overline{x_j} = \lambda\overline{\lambda}y_i\overline{y_j} = y_i\overline{y_j}$ given any $i, j \in [M]$. Likewise, $x \sim \overline{y}$ implies $x_i\overline{x_j} = \overline{y_i}y_j$ for any $i, j \in [M]$. Thus, $x \equiv y$ implies that $\mathrm{Re}(x_i\overline{x_j}) = \mathrm{Re}(y_i\overline{y_j})$ for all $i, j \in [M]$.

For the converse, suppose $x, y \in \mathbb{C}^M$ where $\mathrm{Re}(x_i\overline{x_j}) = \mathrm{Re}(y_i\overline{y_j})$ for all $i, j \in [M]$. Lemma 4.13 implies that $x_i\overline{y_j} = y_i\overline{y_j}$ for all $i, j \in [M]$ or that $x_i\overline{x_j} = \overline{y_i}y_j$ for all $i, j \in [M]$. Suppose first that $x_i\overline{y_j} = y_i\overline{y_j}$ for all $i, j \in [M]$. Then, $|x_k|^2 = |y_k|^2$ for all $k \in [M]$, and in particular $x_k = \lambda_k y_k$ for some $\lambda_k \in \mathbb{T}$. Thus, given $i, j \in [M]$, $x_i\overline{x_j} = \lambda_i\overline{\lambda_j}y_i\overline{y_j}$ and

$$\lambda_i\overline{\lambda_j}y_i\overline{y_j} = y_i\overline{y_j}.$$

If $y_i, y_j \neq 0$ it follows that $\lambda_i\overline{\lambda_j} = 1$ and that $\lambda_i = \lambda_j$. Thus for any indices $i, j$ with $y_i, y_j \neq 0$ we have $x_i = \lambda y_i$ and $x_j = \lambda y_j$ for some $\lambda \in \mathbb{T}$. For any index $k$ with $y_k = 0$ we have that $x_k = 0$ and trivially that $x_k = \lambda y_k$. Therefore, $x_k = \lambda y_k$ for all $k \in [M]$ where $\lambda \in \mathbb{T}$, implying that $x \sim y$.

If we instead suppose that $x_i\overline{x_j} = \overline{y_i}y_j$ we can similarly show that there exists

$\lambda' \in \mathbb{T}$ where $x_k = \lambda' y_k$ for all $k \in [M]$. Hence, $\mathrm{Re}(x_i\overline{x_j}) = \mathrm{Re}(y_i\overline{y_j})$ for all $i, j \in [M]$ implies that $x \sim y$ or $x \sim \overline{y}$. Therefore, $x \equiv y$. □

## 4.3 Conjugate Phase Retrieval on $\mathbb{C}^2$

In this section, we characterize all conjugate phase retrievable frames on $\mathbb{C}^2$.

**Theorem 4.14.** *Any frame over $\mathbb{C}^2$ that is strictly conjugate phase retrievable must be a frame contained in $W$.*

*Proof.* Let $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ be a strictly conjugate phase retrievable frame over $\mathbb{C}^2$. We first write the frame matrix of $\Phi$ as

$$\begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2n} \end{pmatrix} = \begin{pmatrix} | & | & & | \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ | & | & & | \end{pmatrix}$$

By Theorem 4.8, there exists $y = (y_1\ y_2)^T$ in $\mathbb{C}^2$ with $y \notin W$ and

$$\begin{aligned} \mathrm{Im}(\varphi_{11}\overline{\varphi_{21}})\,\mathrm{Im}(y_1\overline{y_2}) &= 0 \\ \mathrm{Im}(\varphi_{12}\overline{\varphi_{22}})\,\mathrm{Im}(y_1\overline{y_2}) &= 0 \\ &\vdots \\ \mathrm{Im}(\varphi_{1n}\overline{\varphi_{2n}})\,\mathrm{Im}(y_1\overline{y_2}) &= 0. \end{aligned}$$

By assumption, $y \not\sim \overline{y}$, and we must have $y_1\overline{y_2} \neq \overline{y_1}y_2 = \overline{y_1\overline{y_2}}$. To satisfy the above list of equations we must then have $\text{Im}(y_1\overline{y_2}) \neq 0$. Thus,

$$\text{Im}(\varphi_{11}\overline{\varphi_{21}}) = \cdots = \text{Im}(\varphi_{1n}\overline{\varphi_{2n}}) = 0.$$

For any frame vector $\varphi_i$, we have $\text{Im}(\varphi_{1i}\overline{\varphi_{2i}}) = 0$, which implies $\varphi_i \in W$. Thus, $\Phi \subseteq W$. Thus, we can say that any strictly conjugate phase retrievable frame over $\mathbb{C}^2$ is a frame in $W$.

□

*Remark.* By Proposition 4.5 and Theorem 4.14,we only need to consider real-valued frame to characterize the strictly conjugate phase retrievable frames in $\mathbb{C}^2$. This implies that a conjugate phase retrievable frame in $\mathbb{C}^2$ containing any complex vectors not in $W$ must also be complex phase retrievable. Thus, the number of vectors required for a conjugate phase retrievable generic frame in $\mathbb{C}^2$ is 4, the same as with complex phase retrieval (See Section 2.2.1,[6]).

Consider a real valued frame $\Phi = \{\hat{a}, \hat{b}, \hat{c}\}$ on $\mathbb{C}^2$, where $\hat{a} = (a_1\ a_2)^T, \hat{b} = (b_1\ b_2)^T, \hat{c} = (c_1\ c_2)^T$. Let $x = (x_1\ x_2)^T, y = (y_1\ y_2)^T$ be vectors in $\mathbb{C}^2$ with

$|\langle x,\varphi\rangle|^2 = |\langle y,\varphi\rangle|^2$ for all $\varphi \in \Phi$. We expand the squared inner products to write

$$\begin{aligned} |\langle x,\varphi\rangle|^2 &= \langle x,\varphi\rangle \overline{\langle x,\varphi\rangle} \\ &= (x_1\varphi_1 + x_2\varphi_2)(\overline{x_1}\varphi_1 + \overline{x_2}\varphi_2) \\ &= \varphi_1^2|x_1|^2 + 2\varphi_1\varphi_2(\mathrm{Re}(x_1\overline{x_2})) + \varphi_2^2|x_2|^2 \end{aligned}$$

.

where $\varphi \in \Phi$.

It follows that $|\langle x,\varphi\rangle|^2 = |\langle y,\varphi\rangle|^2$ if and only if

$$\varphi_1(|x_1|^2 - |y_1|^2) + 2\varphi_1\varphi_2(\mathrm{Re}(x_1\overline{x_2}) - \mathrm{Re}(y_1\overline{y_2})) + \varphi_2(|x_2^2| - |y_2|^2) = 0.$$

Thus, $|\langle x,\varphi\rangle|^2 = |\langle y,\varphi\rangle|^2$ for each $\varphi \in \Phi$ if and only if

$$\begin{aligned} a_1(|x_1|^2 - |y_1|^2) + 2a_1a_2(\mathrm{Re}(x_1\overline{x_2}) - \mathrm{Re}(y_1\overline{y_2})) + a_2(|x_2^2| - |y_2|^2) = 0. \\ b_1(|x_1|^2 - |y_1|^2) + 2b_1b_2(\mathrm{Re}(x_1\overline{x_2}) - \mathrm{Re}(y_1\overline{y_2})) + b_2(|x_2^2| - |y_2|^2) = 0. \\ c_1(|x_1|^2 - |y_1|^2) + 2c_1c_2(\mathrm{Re}(x_1\overline{x_2}) - \mathrm{Re}(y_1\overline{y_2})) + c_2(|x_2^2| - |y_2|^2) = 0. \end{aligned}$$

Let $v_1 = |x_1|^2 - |x_2|^2$, $v_2 = \mathrm{Re}(x_1\overline{x_2}) - \mathrm{Re}(y_1\overline{y_2})$ and $v_3 = |x_2|^2 - |y_2|^2$. Then, we restate the conditions in matrix form:

$$\begin{bmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ b_1^2 & 2b_1b_2 & b_2^2 \\ c_1^2 & 2c_1c_2 & c_2^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, with $v_1, v_2, v_3$ defined as earlier, the frame $\Phi = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ is conjugate phase retrievable if and only if

$$\begin{bmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ b_1^2 & 2b_1b_2 & b_2^2 \\ c_1^2 & 2c_1c_2 & c_2^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

implies $v_1 = v_2 = v_3 = 0$. In other words, with $A = \begin{bmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ b_1^2 & 2b_1b_2 & b_2^2 \\ c_1^2 & 2c_1c_2 & c_2^2 \end{bmatrix}$, $\Phi$ is conjugate phase retrievable if and only if $\ker A = \{\hat{0}\}$.

**Theorem 4.15.** *A frame* $\Phi = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$ *is conjugate phase retrievable on* $\mathbb{C}^2$ *if and only if the matrix*

$$\Psi = \begin{bmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ b_1^2 & 2b_1b_2 & b_2^2 \\ c_1^2 & 2c_1c_2 & c_2^2 \end{bmatrix}$$

*is invertible.*

As a consequence of Theorem 4.15, we derive the following characterization of conjugate phase retrievable real-valued frames in $\mathbb{C}^2$ :

**Theorem 4.16.** *A real-valued frame $\Phi \subseteq \mathbb{C}^2$ is conjugate phase retrievable if and only if $\Phi$ has the complement property.*

*Proof of Theorem 4.15.* First, suppose that $\Psi$ is invertible. Then, $\ker \Psi$ is trivial, namely $\Psi v = 0$ if and only if $v = 0$. For any $x, y \in \mathbb{C}^2$ and

$$v = \begin{bmatrix} |x_1|^2 - |y_1|^2 \\ \mathrm{Re}(x_1\overline{x_2}) - \mathrm{Re}(y_1\overline{y_2}) \\ |x_2|^2 - |y_2|^2 \end{bmatrix},$$

$\Psi(v) = 0$ then implies by Lemma 4.12 that $x \sim y$ or $x \sim \overline{y}$. Further, $\Psi v = 0$ if and only if $x, y$ have the same phaseless measurements with $\Phi$. Hence, $\Phi$ is conjugate phase retrievable on $\mathbb{C}^2$.

Conversely, with a computation in Sage we receive

$$\det \Psi = -2(a_1b_2 - a_2b_1)(a_1c_2 - a_2c_1)(b_1c_2 - b_2c_1).$$

Then, $\det \Psi = 0$ if and only if one of the following equations hold:

$$a_1b_2 = a_2b_1$$

$$a_1c_2 = a_2c_1$$

$$b_1c_2 = b_2c_1.$$

The first equation occurs exactly when $a$ and $b$ are linearly dependent, and the second occurs exactly when $a$ and $c$ are linearly dependent. Similarly, the third question occurs if and only if $b, c$ are linearly dependent. Since $\Phi$ has only 3 vectors, we can say that $\Psi$ has the complement property if and only if any two of $a, b, c$ are linearly independent. Thus, $\Phi$ has the complement property if and only if $\det \Psi \neq 0$.

Suppose $\Phi$ is conjugate phase retrievable. Then, Theorem 4.11 implies that $\Phi$ has the complement property and further that $\det \Psi \neq 0$. Hence, $\Psi$ is invertible. □

By combining Theorem 4.16 and Theorem 4.14 we can conclude the following about the minimal number of vectors required for conjugate phase retrieval in $\mathbb{C}^2$ :

**Theorem 4.17.** *1. Any frame $\Phi \not\subseteq W$ on $\mathbb{C}^2$ that is conjugate phase retrievable must have $|\Phi| \geq 4$, which is a sharp lower bound.*

*2. Any real-valued frame $\Phi$ on $\mathbb{C}^2$ that is conjugate phase retrievable must have $|\Phi| \geq 3$, where 3 is a sharp lower bound.*

*Proof.* 1. Suppose $\Phi$ is conjugate phase retrievable and $\Phi \not\subseteq W$. By Theorem 4.14, $\Phi$ is not strictly conjugate phase retrievable, so we must have that $\Phi$ is complex phase retrievable on $\mathbb{C}^2$. In [6] it was proven that a minimum of 4 vectors is required for complex phase retrieval on $\mathbb{C}^2$ and is a sharp lower bound.

2. Suppose $\Phi$ is conjugate phase retrievable and $\Phi$ consists of all real vectors in $\mathbb{C}^2$. Then, $\Phi \subseteq W$ and Theorem 4.16 implies that $\Phi$ has the complement property. Hence, $|\Phi| \geq 2(2) - 1 = 3$. Example 4.1 gives a conjugate phase

retrievable real-valued frame of 3 vectors and thus shows $|\Phi| \geq 3$ is a sharp lower bound.

$\square$

## 4.4 Discussion and Open Questions

In the final section of this thesis, we propose several interesting open questions concerning unsigned sampling in Paley-Wiener spaces and conjugate phase retrieval in Chapters 3 and 4. -

**Conjugate Phase Retrieval in $\mathbb{C}^M$:**

1. What frames are strictly conjugate phase retrievable in $\mathbb{C}^M$ for $M \geq 3$? Is it true that these frames are in $W$?

2. Theorem 4.17 implies that the minimum generic number of vectors for real-valued frames to be conjugate phase retrievable in $\mathbb{C}^2$ is 3. What is the minimal number of vectors for a generic *real-valued* frame to be conjugate phase retrievable in $\mathbb{C}^M$?

3. Again in light of Theorem 4.17, what is the number of vectors required for a generic complex frame on $\mathbb{C}^M$ to be conjugate phase retrievable? Generic complex frames on $\mathbb{C}^2$ are not in $W$. Thus, Theorem 4.17 implies that given a generic frame on $\mathbb{C}^2$, the minimum number of vectors for $\Phi$ to be conjugate phase retrievable is 4 vectors, the same as for complex phase retrieval. If it

is true that strictly conjugate phase retrievable frames in $\mathbb{C}^M$ are in $W$, then we can argue as in Theorem 4.17 that the number of generic complex vectors needed for conjugate phase retrieval is exactly the number of generic complex vectors needed for complex phase retrieval in $\mathbb{C}^M$.

$PW$ **Sets of Uniqueness**: Question 1 on page 34 is currently open. Is it true that $\Lambda$ is a USS for ${}_{\mathbb{R}}PW^2_\Omega$ if and only if $\Lambda$ is a set of uniqueness for $PW^1_{\Omega+\Omega}$? Questions 2 and 3 from pages 38-40 are also open and can be investigated by the following: is there some exact set of sampling of $PW^2_1$ that is no longer a USS for ${}_{\mathbb{R}}PW^2_{\frac{1}{2}}$ after the removal of an element? Note that none of the above questions have been answered for general $\Omega$ or even for considering $\Omega$ as an interval.

**Conjugate Unsigned Sampling**: Ultimately, we wish to perform conjugate unsigned sampling on Paley-Wiener spaces. Can we find $\Lambda \subseteq \mathbb{R}$ that is a set of conjugate unsigned sampling on $PW^2_1$?

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