ALGEBRAIC AND COMBINATORIAL ASPECTS OF POLYTOPES AND DOMINO TILINGS



A thesis presented to the faculty of
San Francisco State University
In partial fulfillment of
The Requirements for
The Degree

Master of Arts
In
Mathematics

by

Nicole Yamzon

San Francisco, California

February 2019

CERTIFICATION OF APPROVAL

I certify that I have read *ALGEBRAIC AND COMBINATORIAL ASPECTS OF POLYTOPES AND DOMINO TILINGS* by Nicole Yamzon and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

Federico Ardila
Professor of Mathematics

Serkan Hosten
Professor of Mathematics

Elizabeth Gross
Assistant Professor of Mathematics

ALGEBRAIC AND COMBINATORIAL ASPECTS OF POLYTOPES AND DOMINO TILINGS

Nicole Yamzon
San Francisco State University
2019

Two classical objects of study in combinatorics are polytopes and domino tilings. In the 1990s William Thurston proved that the set of domino tilings of a simply-connected region is connected by flips. The first part of this thesis will introduce definitions associated to the underlying grid graph of a collection of domino tilings. We will then provide an analogous proof of Thurston's result by using the language of toric ideals.

The second half of this thesis, a joint project with Anastasia Chavez, focuses on $d$-dimensional simplicial polytopes. In particular, we introduce the associated face vector i.e. the $f$-vector of a $d$-dimensional polytope. In conclusion, we prove a correspondence between the maximal linearly independent subsets of the $f$-vectors of simplicial polytopes and the Catalan numbers.

I certify that the Abstract is a correct representation of the content of this thesis.

Federico Ardila M — March 21, 2019

Chair, Thesis Committee — Date

# ACKNOWLEDGMENTS

I would like to express my very great appreciation to my wonderful advisors, Dr. Federico Ardila and Dr. Elizabeth Gross.

Gratitude goes out to the Mathematics Departments at San Francisco State University and San José State University. In particular, thank you to my committee.

To all that made this possible. You know who you are.

TABLE OF CONTENTS

LIST OF FIGURES

# Chapter 1

# Domino Tilings

## 1.1 Background

The origins of tilling dominoes goes as far back as the 13th century mostly arising from games of strategy. Since the 1960's the formalization of dominoes and the regions they cover have become relevant to a variety of mathematical branches. In 1961, Wang presented a problem in mathematical logic framed around deciding a region could be tiled or not [24]. In 1966 Wang's student Berger reinterpreted the solution to Wang's question into a version of the *halting problem* by connecting Turing Machines to tilings [2]. In addition to Wang's result in 1961 Kasteleyn and Fisher–Temperley proved independently that

$$4^{mn}\prod_{j=1}^{m}\prod_{k=1}^{n}\left(\cos^2\frac{2\pi}{2m+1}+\cos^2\frac{k\pi}{2n+1}\right)$$

enumerates the number of non-overlapping domino tilings of a $2m \times 2n$ rectangle. Discovering this precise formula was motivated entirely by research in phase transitions of matter governed by the laws of thermodynamics[12, 10]. The Kasteleyn and Fisher–Temperley result demonstrates that enumeration of domino tilings can have meaningful interpretations in other scientific areas.

Wang and Berger's results are illustrative of the inherent complexity underlying the space of domino tilings.

In the first half of this thesis, our focus will be on dominoes and tilings. The study of dominoes and tilings goes back to the 1960s and has been of much recent interest [11].

**Definition 1.1.** A *domino* is formed by two unit squares connected along a complete edge.

**Definition 1.2.** A *domino tiling* of a region $R$ is defined to be a covering of $R$ with dominoes such that each domino covers exactly two squares of the region and no two dominoes overlap.

The pertinent topic of the first half of this thesis will be the connectivity of the space of tilings of a region. The main mathematical tool used to analyze the underlying structure of connectivity will be the theory of toric ideals. We will focus on the world of 2-dimensional domino tilings. In terms of connectivity, various methods have been used to characterize the types of "local moves" from one tiled region to another. One of these moves is called a *local flip*, or equivalently, a *flip*.

**Definition 1.3.** A *local flip* is the simplest move where we remove a pair of two adjacent parallel dominoes and replace them with two adjacent parallel dominoes in a perpendicular direction to the first pair.

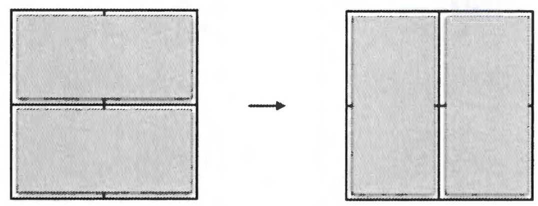


Figure 1.1: A local flip

**Definition 1.4.** We say two tilings $T_1$ and $T_2$ are *flip connected* if there exists a sequence of local flips where we can obtain $T_2$ from $T_1$. A cubiculated region $R$ is *flip connected* if every two tilings of $R$ are flip connected.

In 1990, Thurston gave a set of necessary conditions on $R$ in order for the space of tilings of $R$ to be flip connected.

**Theorem 1.5.** *[22] If $R$ is a simply connected region in 2-dimensions, then $R$ is flip-connected.*

In this thesis, we give an alternative proof of Theorem 1.5 using algebraic methods.

## 1.2 Connections to graph theory

It is helpful to think of tilings of a cubiculated region as perfect matchings of a graph. Here we set up the terminology to construct this correspondence.

**Definition 1.6.** Given a graph $G = (V, E)$, a *matching* $M$ is an independent edge set, i.e. no two edges of $M$ share a vertex. A *perfect matching* is a matching that covers all vertices in $G$.

**Definition 1.7.** Let $R$ be a two-dimensional region. We define the *graph associated to the domino tiling of* $R$ be the graph that has one vertex for each square in $R$ and has an edge between a pair of vertices if their two corresponding squares in $R$ are connected by a line segment. We will denote this graph by $G_R$.

**Definition 1.8.** Let the region $R$ be the $m \times n$ box. Then $G_R$ is denoted by $G_{m,n}$ and is the $m \times n$ grid graph whose vertices correspond to the points in $[0, m] \times [0, n] \cap \mathbb{Z}^2$.

By the construction of $G_R$ from $R$, we can see that there is a one-to-one correspondence between tilings of $R$ and perfect matchings on the graph $G_R$. This correspondence is illustrated in the example below.

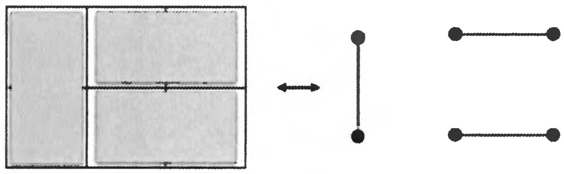


Figure 1.2: Let $R = B_{2,3}$, the $2 \times 3$ box. We have a tiling on $R$ and a perfect matching of $G_{2,3}$

For the rest of this paper, we will refer to tilings and matchings interchangeably. We can use the underlying graph structure to be able to understand the space of 2-dimensional domino tilings. While tilings on a cubiculated region $R$ can be characterized by perfect matchings on $G_R$, moves between two tilings can be characterized by even cycles.

**Definition 1.9.** A *walk* on $G$ is a finite sequence of the form

$$w = \big((v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\big)$$

with each $v_i \in V(G)$ and $(v_{i-1}, v_i) \in E(G)$. In the case where $v_1 = v_n$ then $w$ is called a *closed walk*. A *cycle* is a closed walk that traverses each vertex in the walk exactly once. The *length* of a cycle or closed walk is the number of edges in the walk. A closed walk is *even* if the cycle has even length. An even closed walk is *primitive* if it does not contain a proper closed even subwalk.

**Proposition 1.10.** *Let $R$ be a two-dimensional cubiculated region. Every cycle of the graph $G_R$ is even.*

*Proof.* Notice that for any two dimensional cubiculated region $R$, the graph $G_R$ is a subgraph of $G_{m,n}$ for some $m$ and $n$. In particular, $G_R$ is bipartite. Thus, every cycle of $G_R$ is even [25]. $\square$

*Remark.* Since every cycle of $G_R$ is even, the only primitive even closed walks on $G_R$ are cycles. [15]

In the following proposition, we see that the union of two tilings of $R$ corresponds to a collection of cycles on $G_R$.

**Proposition 1.11.** *Let $T_1$ and $T_2$ be tilings and let $G = T_1 \cup T_2$ (considered as a multigraph). Then $G$ will be a disjoint collection of even cycles. Note: We will be considering 2-cycles in our collection.*

*Proof.* Consider the graph $G = T_1 \cup T_2$ with $n$ vertices. We know for each $i \in V(G)$ the $deg_G(i) = 2$. By definition $G$ must be a 2-regular graph of size $n$. A characterization of 2-regular graphs gives us that $G$ will be formed by a disjoint collection of cycles which is what we wanted to show. $\square$

Since $G$ is a disjoint collection of cycles we introduce the following terminology.

**Definition 1.12.** Let $G$ be a graph. We say $\mathcal{C} = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r\}$ is a *cycle cover* of $G$ if each $\mathcal{C}_i$ is a cycle and every vertex in $G$ is covered by exactly one $\mathcal{C}_i$.

Note that given a region $R$ and two tilings, $T_1$ and $T_2$, the multigraph $\mathcal{C} = T_1 \cup T_2$ is a cycle cover of $G_R$. Additionally, we can think of the edges in $\mathcal{C}$ as two-colorable, specifically, we can color the edges corresponding to $T_1$ red and the edges corresponding to $T_2$ as blue.

Finally, we end this section with a discussion on chords, which will play a role in the algebra in the next two sections.

**Definition 1.13.** Let $C$ be a cycle of a graph $G = (V, E)$. An edge $e \in E$ is a *chord* of $C$ if $e$ connects two vertices covered by $C$, but is not in $C$. A cycle is *chordless* if it does not have a chord in $G$.

**Definition 1.14.** Let $C = \big((v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)\big)$ be an even cycle of a graph $G = (V, E)$. Let $e = (v_i, v_j) \in E$ be a chord of $C$ with $i < j$. We say $e$ is an *even chord* if $j - i$ is odd, in other words, if the two new cycles obtained by adding $e$ to $C$ are both even.

*Remark.* Note that since every cycle in $G_R$ is even, every chord of a cycle in $G_R$ is even.

## 1.3 Toric Ideals of Graphs and Tiling Ideals

Given a graph there are several algebraic structures we can describe in association to that graph. While discussing the connection between domino tilings and cycles of grid graphs we will restrict to the realm of *toric ideals.*

### 1.3.1 Toric Ideal of Graphs

We start the discussion for toric ideals by defining a fundamental algebraic object called a *ring*.

**Definition 1.15.** Let $R$ be a set equipped with two binary operations $(+, \cdot)$, which refer to as addition and multiplication, respectively. We say $R$ forms a ring when the operations satisfies the following axioms:

(I) $R$ forms an abelian group under addition.

(II) Multiplication is associative.

(III) Multiplication is distributive with respect to addition.

**Example 1.1.** Let $R = \mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ and our binary operations the usual $(+, \cdot)$. One can verify that $R$ forms a ring with a multiplication table and Definition 1.15

Rings in a general setting have lots of underlying structure. One of these underlying objects is an *ideal.*

**Definition 1.16.** We define an ideal of a ring $R$ as a subset $I \subseteq R$ that satisfies the following axioms:

(I) If $r, s \in I$ then $r + s \in I$.

(II) If $r \in R$ and $s \in I$ then $r \cdot s \in I$.

An analogous structure in Group Theory is normal subgroups of a group. A key property that connects these two algebraic objects is that we can think of $I_R$ as a kernel of a ring homomorphism from $R$ similar to the way we can describe normal subgroups of $G$ as the kernel of a group homomorphism.

In order to apply the theory of toric ideals we will talk about a particular family of rings.

**Definition 1.17.** Let $\mathbb{K}$ be a field. A *polynomial ring* $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, x_2, \ldots, x_n]$ in $n$-variables is a vector space over $\mathbb{K}$ with respect to addition and multiplication by a scalar.

The toric ideals that we will study are ideals of polynomial rings. One property of polynomial rings that will be used is the fact that their ideals are finitely generated [9].

**Definition 1.18.** Let $\mathbb{K}[\mathbf{x}]$ be a polynomial ring. Let $I \subseteq \mathbb{K}[\mathbf{x}]$ be an ideal of $\mathbb{K}[\mathbf{x}]$. Let $\{g_0, g_1, \ldots, g_k\} \subseteq I$. We say $I$ is *finitely generated* by $\{g_0, g_1, \ldots, g_k\}$ if for all $r \in I$ we have

$$r = \sum_{i=0}^{k} \alpha_i g_i,$$

where $\alpha_i \in \mathbb{K}[\mathbf{x}]$. Then we call $\{g_0, g_1, \ldots, g_k\}$ a *generating set* of $\mathbb{K}[\mathbf{x}]$.

Now we are ready to define *toric ideal of a graph.*

**Definition 1.19.** Let $G$ be a graph with edge set $E$ and vertex set $V$. Let $\mathbb{K}[E] = \mathbb{K}[y_e \mid e \in E[G]$ and $\mathbb{K}[V] = \mathbb{K}[x_v \mid v \in V(G)]$.

We construct an associated homomorphism of $G$ as follows:

$$\phi_G : \mathbb{K}[E] \to \mathbb{K}[V]$$

$$y_{e_{i,j}} \mapsto x_i x_j$$

Then the *toric ideal* of $G$, denoted $I_G$, is

$$I_G := \ker(\phi_G) = \{f \in \mathbb{K}[E] \ : \ \phi_G(f) = 0\}.$$

Toric ideals of graphs have been well-studied, and in fact, their generating sets can be described by primitive closed even walks.

**Definition 1.20.** Let $w$ be a closed even walk,

$$w = \Big((v_1, v_2), (v_2, v_3), \dots, (v_{2n}, v_1)\Big),$$

where $e_i = (v_1, v_{i+1})$. Then we can define a binomial arising from closed even walk

$$B_w = \prod_{i=1}^{n} y_{e_{2i-1}} - \prod_{i=1}^{n} y_{e_{2i}}.$$

**Proposition 1.21.** *[23] Given graph $G$, the ideal $I_G$ is generated by the set of* *binomials arising from closed even primitive walks on $G$.*

Just as we can define a binomial arising from a closed walk, we can define a binomial associated to two tilings.

**Definition 1.22.** Given two tilings $T_1$ and $T_2$ of a cubiculated region $R$ regarded as perfect matchings in $G_R$, we can define the binomial arising from $(T_1, T_2)$ to be

$$B_{T_1,T_2} = \prod_{e_i \in T_1} y_{e_i} - \prod_{e_j \in T_2} y_{e_j}.$$

**Definition 1.23.** For ease of notation, we will use the following monomial shorthand. Let $E_0 \subseteq E(G)$, then we define

$$y^{E_0} := \prod_{e_i \in E_0} y_{e_i}.$$

Thus, we will write $B_{T_1,T_2}$ as

$$B_{T_1,T_2} = y^{T_1} - y^{T_2}.$$

**Proposition 1.24.** *Let $R$ be a cubiculated region. Let $T_1$ and $T_2$ be two tilings of $R$. Then the binomial $B_{T_1,T_2}$ arising from $(T_1, T_2)$ is in the toric ideal $I_{G_R}$.*

*Proof.* Let $T_1$ and $T_2$ be two tilings and $B_{T_1,T_2}$ be the associated binomial. We want to show that $B_{T_1,T_2} \in I_{G_R}$. Consider the image

$$\phi_{G_R}(B_{T_1,T_2}).$$

Then by definition of the binomial we get

$$\phi_{G_R}(B_{T_1,T_2}) = \phi_{G_R}(y^{T_1} - y^{T_2}).$$

Then since $\phi_{G_R}$ is a homomorphism we get to distribute and get

$$\phi_{G_R}(B_{T_1,T_2}) = \phi_{G_R}(y^{T_1}) - \phi_{G_R}(y^{T_2}) = \prod_{e_i \in T_1} \phi_{G_R}(y_{e_i}) - \prod_{e_j \in T_2} \phi_{G_R}(y_{e_j}).$$

Then by definition of $\phi_{G_R}$ and since $T_1$ and $T_2$ are both perfect matchings of $G_R$

$$\phi_{G_R}(B_{T_1,T_2}) = \prod_{i \in V(G_R)} x_i - \prod_{i \in V(G_R)} x_i = 0.$$

Therefore, we get cancellation of the monomials and $B_{T_1,T_2} \in I_{G_R}$. □

### 1.3.2 Moves between tilings

We are concerned with moving between two tilings $T_1$ and $T_2$ of a cubiculated region. In essence, a move is replacing a set of dominoes $D_1$ with another set of dominoes $D_2$, such that the replacement results in a tiling. We will encode a move by $(D_1, D_2)$.

A move has size $d$ if $|D_1| = |D_2| = d$.

**Definition 1.25.** Given a cubiculated region $R$ and associated graph $G_R$. The *binomial arising from the tiling move* $(D_1, D_2)$ in $I_{G_R}$ is

$$y^{D_1} - y^{D_2}.$$

**Theorem 1.26.** *If $I_{G_R}$ is generated by binomials of degree d or less, then the set of tilings of a cubiculated region R is connected by moves of size d or less.*

*Proof.* Suppose $I_{G_R}$ is generated by binomials of degree less than or equal to $d$.

Let $U$, $V$ be two tilings of $R$. Let $\mathcal{A} = \{y^{u_1} - y^{v_1}, \ldots, y^{u_k} - y^{v_k}\}$ be a generating set of $I_{G_R}$ where $\deg(y^{u_i} - y^{v_i}) \leq d$ for each $1 \leq i \leq k$. Then $B_{U,V} = y^U - y^V = \alpha_1(y^{u_1} - y^{v_1}) + \ldots + \alpha_k(y^{u_k} - y^{v_k})$ where $\alpha_i \in \mathbb{K}[E(G_R)]$ for all $i$. Notice that, by distributively, we can rewrite $B_{U,V}$ as follows

$$B_{U,V} = \alpha_{i_1}(y^{u_{i_1}} - y^{v_{i_1}}) + \ldots + \alpha_{i_r}(y^{u_{i_r}} - y^{v_{i_r}})$$

where $y^{u_{i_j}} - y^{v_{i_j}} \in \mathcal{A}$ and each $\alpha_{i_j}$ is a non-zero monomial. We will specify an order in which to write our $\alpha_i$'s. Since the left hand side contains $y^U$ lets choose $i_1$ such that

$$y^U = \alpha_1 y^{u_{i_1}}.$$

and for each subsequent $j$ choose $i_{j+1}$ such that

$$\alpha_{j+1}y^{u_{i_{j+1}}} = \alpha_j y^{v_{i_j}}.$$

The arrangement implies that $B_{U,V}$ is a telescoping sum starting with $y^U$ and ending with $y^V$. Using the binomial expansion of $B_{U,V}$ in terms of $\mathcal{A}$, let us construct a sequence of tilings that correspond to the intermediate terms of our telescoping sum:

$$T_0 = U$$

$$T_1 = (T_0 \setminus u_{i_1}) \cup v_{i_1}$$

$$T_2 = (T_1 \setminus u_{i_2}) \cup v_{i_2}$$

$$\vdots$$

$$T_r = (T_{r-1} \setminus u_{i_r}) \cup v_{i_r}) = V$$

and in general

$$T_j = (T_{j-1} \setminus u_{i_j}) \cup v_{i_j}$$

By construction, ,the move that takes us from $T_{j-1}$ to $T_j$ is $(u_{i_j}, v_{i_j})$. Since $\deg(y^{u_{i_j}} - y^{v_{i_j}}) \leq d$, the move $(u_{i_j}, v_{i_j})$ has size less than or equal to $d$, thus $U$ and $V$ are connected by moves of size $d$ or less.

□

**Theorem 1.27.** *Let $R$ be a cubiculated region with the associated graph $G_R$. Then $R$ is connected by the set of moves corresponding to the set of chordless cycles of $G_R$.*

*Proof.* By Lemma 3.1 in [15] $I_{G_R}$ is generated by the set of binomials arising from primitive closed even walks on $G_R$. Since $G_R$ is bipartite, the only primitive closed even walks on $G_R$ are even cycles (see, for example, Proposition 2.2 in [16]) . Furthermore, by the work of Ohsugi and Hibi , we can further reduce the generating set to binomials arising from closed even chordless cycles of $G_R$[15]. Finally, from the Proof of 1.26, we see that the set of moves corresponding to the binomials in a generating set of $I_{G_R}$ will connect the space tilings of $R$. □

**Corollary 1.28.** *Let $2d$ be the size of the largest even chordless cycle in $G_R$. Then the space of tilings of $R$ is connected by moves of size $d$ or less.*

### 1.3.3 Tiling Ideals

Theorem 1.27 and Corollary 1.3.2 give a bound on the size of moves need to connect the space of tilings of a region $R$, however, this bound can be arbitrarily large. For example, let $m, n \geq 3$, then $G_{m,n}$ contains a chordless cycle of length $2(m+n)$.

A flip move $(D_1, D_2)$ corresponds to a 4-cycle in $G_R$ and the corresponding binomial $y^{D_1} - y^{D_2}$ has degree 2. Conversely, any quadratic binomial in $I_{G_R}$ must correspond to a 4-cycle [16], and consequently, a flip move. Thus, to show that the space of tilings of a region $R$ is flip connected, we need to show that every binomial arising from two tilings is generated by quadratics.

**Definition 1.29.** Let $R$ be a cubiculated region with associated graph $G_R$. The *flip ideal* of $R$ is defined as follows:

$$I_{R_{flip}} := \langle y^{D_1} - y^{D_2} \ : \ (D_1, D_2) \text{ is a flip move } \rangle \subseteq I_{G_R}.$$

The *tiling ideal* of $R$ is defined as follows:

$$I_{R_{tiling}} = \langle y^{T_1} - y^{T_2} \ : T_1,\ T_2 \text{ are tilings of R } \rangle \subseteq I_{G_R}.$$

Using the flip and tiling ideal, we can use the language of ideals to restate what it means for a region to be flip-connected.

**Proposition 1.30.** *A region $R$ is flip-connected if and only if*

$$I_{R_{tiling}} \subseteq I_{R_{flip}}.$$

*Proof.* Using the correspondence between binomials and moves established in the proof of Theorem 1.26, we can see a binomial arising from two tilings $B_{T_1,T_2}$ is generated by quadratics if and only if $T_1$ and $T_2$ are connected by moves of size 2. The only moves of size two are flip moves. $\square$

### 1.3.4 Lemmas on binomials arising from tilings

In this section, we state and prove a couple of lemmas that will be helpful in giving an algebraic proof of Theorem 1.5.

**Lemma 1.31.** *Let $T_1$ and $T_2$ be two tilings of a cubiculated region $R$ with corresponding cycle cover $T_1 \cup T_2 = \mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_r\}$ of $G_R$. Then $B_{T_1,T_2}$ can be written as a linear combination of binomials arising from walks on $\mathcal{C}_1, \dots, \mathcal{C}_r$.*

*Proof.* We will induct on $r$, the number of cycles in $\mathcal{C}$.

Construct a closed walk $w_i$ on each $\mathcal{C}_i \in \mathcal{C}$ by starting with an edge in $T_1$, then the binomial arising from $w_i$ is as follows:

$$B_{\mathcal{C}_i} := \prod_{e \in T_1 \cap \mathcal{C}_i} y_e - \prod_{e \in T_2 \cap \mathcal{C}_i} y_e.$$

Next notice that the a binomial arising from a 2-cycle $\mathcal{C}_i$ has the following form

$$y_j - y_j.$$

Therefore adding $B_{\mathcal{C}_i}$ is equivalent to adding zero. Thus, we will assume that each cycle in $\mathcal{C}$ is of length at least four.

Let $r = 1$. Then $\mathcal{C} = \{\mathcal{C}_1\}$, and consequently,

$$B_{T_1,T_2} = B_{\mathcal{C}_1}.$$

Now, assume the statement holds for $r-1$. Let us construct the following sets:

$$T_1' = T_1 \setminus \{\mathcal{C}_1 \cap T_1\}$$

and

$$T_2' = T_2 \setminus \{\mathcal{C}_1 \cap T_2\}.$$

Notice

$$B_{T_1,T_2} = y^{T_1} B_{\mathcal{C}_1} + y^{\mathcal{C}_1 \cap T_2} B_{T_1',T_2'}.$$

Since $T_1'$ and $T_2'$ are tilings of a cubiculated subregion $R' \subseteq R$ with cycle cover $\{\mathcal{C}_2, \ldots, \mathcal{C}_r\}$, we can apply our inductive hypothesis, thus $B_{T_1',T_2'}$ can be written as a linear combination of binomials arising from $\{\mathcal{C}_2, \ldots, \mathcal{C}_r\}$, and our statement holds. □

**Example 1.2.** Consider the tilings $T_1$ and $T_2$. Let $G = T_1 \cup T_2$ shown in Figure 1.3.

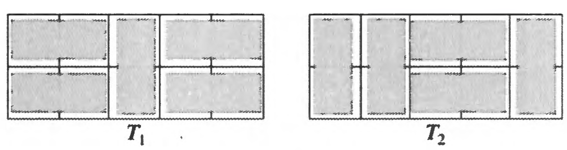



Figure 1.3: $T_1$ and $T_2$

Lets consider a new labeling on $G = \{y_1, y_2, \ldots, y_{10}\}$ where $y_{2i-1} \in T_1$ and

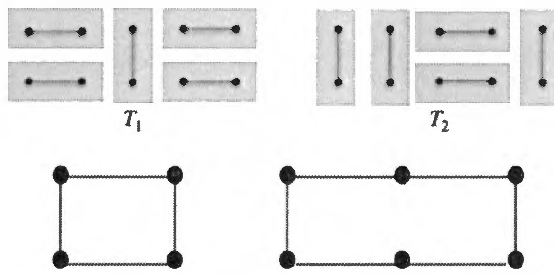



Figure 1.4: $T_1$ and $T_2$ correspond to perfect matchings of the $2 \times 5$-grid.

$y_{2i} \in T_2$. Consider the binomial associated with $G$.

$$B_G = y_1y_3y_5y_7y_9 - y_2y_4y_6y_8y_{10}.$$

We can rewite $B_G$ as follows

$$B_G = y_5y_7y_9(y_1y_3 - y_2y_4) + y_2y_4(y_5y_7y_9 - y_6y_8y_{10}).$$

Notice that $y_5y_7y_9(y_1y_3 - y_2y_4) =$ 4cycle and $y_2y_4(y_5y_7y_9 - y_6y_8y_{10}) =$ 6cycle.

Specific chords in $G_R$, called flip chords, give us hints on when we can write a binomial arising from two tilings in terms of two binomials of smaller degree.

**Definition 1.32.** Let $\mathcal{C}_0$ be a cycle. A *flip chord* of $\mathcal{C}_0$ in $G = (V, E)$ is an edge $e \in E$ such that the graph $\mathcal{C}_0 \cup \{e\}$ contains a 4-cycle. If $\mathcal{C} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_r\}$ is a collection of cycles, then we say $e$ is a *flip chord of* $\mathcal{C}$ *in* $G$ if $e$ is a flip chord of some $\mathcal{C}_i$ in $G$.

**Definition 1.33.** Let $R$ be a two-dimensional cubiculated region and $\mathcal{C}$ a cycle cover of $G_R$. We say $e \in E(G_R)$ is a *boundary flip chord* of $\mathcal{C}$ if $e$ is a flip chord of $\mathcal{C}$ and if the edge opposite $e$ in the constructed 4-cycle is a boundary edge of $G_R$ (an edge $(u, v)$ is an boundary edge of $G_R$ if the squares corresponding to $u$ and $v$ are on the boundary of $R$).

**Proposition 1.34.** *Let $R$ be the disjoint union of a set of 2-dimensional contractible cubiculated regions with associated graph $G_R$. Let $T_1$ and $T_2$ be two tilings of $R$. If $G_R$ contains an edge that is a boundary flip chord of $\mathcal{C} = T_1 \cup T_2$, then $y^{T_1} - y^{T_2}$ can be written as the linear combination of a quadratic and a binomial arising from two tilings $T_1'$ and $T_2'$ of a proper subregion $R' \subset R$ that is the disjoint union of contractible regions.*

*Proof.* Let $R$ be the disjoint union of a set of 2-dimensional contractible cubiculated regions with associated graph $G_R$. Let $T_1$ and $T_2$ be tilings of $R$ and let $\mathcal{C} = T_1 \cup T_2$ be the corresponding cycle cover of $G_R$. Suppose $\mathcal{C}$ has a boundary flip chord in $G_R$. Then, by definition, there exists an edge, which we will denote be $e$ such that $\mathcal{C} \cup \{e\}$ contains a 4-cycle.

Let's label the 4-cycle of $\mathcal{C} \cup \{e\}$ as $(1, 2, 3, e)$ where $e$ is the flip-chord and the edge labeled 2 is on the boundary of $G_R$. Refer to Figure 1.3 and Figure 1.3 for a visual reference of a flip chord. Let $R'$ be the subregion of $R$ obtained by removing the two squares corresponding to the vertices in the edge labeled 2; the subregion $R'$ is the disjoint union of contractible regions since edge 2 is on the boundary of $R$.

In the case where $\{1,3\} \subseteq T_1$, consider the following tilings of $R'$:

$$T_1' = (T_1 \setminus \{1,3\}) \cup \{e\} \qquad T_2' = T_2 \setminus \{2\}.$$

Then can write $y^{T_1} - y^{T_2}$ explicitly in the following form

$$y^{T_1} - y^{T_2} = y^{T_1 \setminus \{1,3\}}(y_1 y_3 - y_2 y_e) + y_2(y^{T_1'} - y^{T_2'}).$$

Thus, we see that $y^{T_1} - y^{T_2}$ can be written in terms of a quadratic binomial associated to the 4-cycle and a binomial arising from the tilings $T_1'$ and $T_2'$ of the proper subregion $R' \subset R$.

Indeed, we observe that

$$|T_1'| = |(T_1 - \{1,3\} \cup \{e\})| = |T_1| - 1 < |T_1|$$

and

$$|T_2'| = |T_2 - \{2\}| = |T_2| - 1 < |T_2|.$$

In the case, where $\{1,3\} \subseteq T_2$, consider the following tilings of $R'$:

$$T_1' = T_1 \setminus \{2\} \qquad T_2' = (T_2 \setminus \{1,3\}) \cup \{e\}.$$

Then can write $y^{T_1} - y^{T_2}$ explicitly in the following form

$$y^{T_1} - y^{T_2} = y_2(y^{T_1'} - y^{T_2'}) + y^{T_2 \setminus \{1,3\}}(y_2 y_e - y_1 y_3).$$

Thus, again, we see that $y^{T_1} - y^{T_2}$ can be written in terms of a quadratic binomial associated to the 4-cycle and a binomial arising from the tilings $T_1'$ and $T_2'$ of the proper sub-region $R' \subset R$.

□

**Example 1.3.** Consider the $2 \times 5$-grid graph. The labeling convention for the edges will be to label the left-north most edge 1 and then label increasing by one in the clock-wise direction.

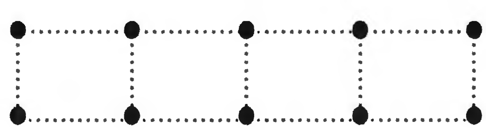
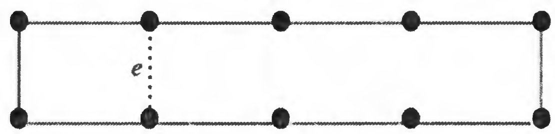


Figure 1.5: $2 \times 5-$grid graph

Suppose $G_k$ contains an edge that is a flip chord. We will call the flip chord $e$. Then by definition of flip chord this implies that given $\mathcal{C}_{T_1,T_2}$ for two tilings we know that $\mathcal{C}_{T_1,T_2} \cup e$ contains a 4-cycle. Consider the binomial formed by $T_1$ and $T_2$,

$$B_{T_1,T_2} = y^{T_1} - y^{T_2}.$$

where explicitly $y^{T_1} = y_1y_3y_5y_7y_9$ and $y^{T_2} = y_2y_4y_6y_8y_{10}$. Consider $\mathcal{C}_{T_1,T_2} \cup \{e\}$,



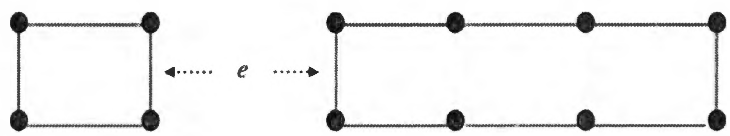




Figure 1.6: The formed cycles, a 4-cycle and and 8-cycle.

We want to show that there exists monomials $\alpha$ and $\beta$ such that $\alpha B_{4\text{cycle}} + \beta B_{8\text{cycle}} = y^{T_1} - y^{T_2}$. We can algebraically solve for $\alpha$ and $\beta$ by using the following expression

$$\alpha(y_1y_9 - y_ey_{10}) + \beta(y_ey_3y_5y_7 - y_2y_4y_6y_8)$$

to choose our $\alpha$ and $\beta$.

Thus, $\alpha = y_3y_5y_7y_9$ and $\beta = y_{10}$. Then we can verify algebraically that

$$y^{T_1} - y^{T_2} = (y_3y_5y_7y_9)B_{4\text{cycle}} + (y_{10})B_{8\text{cycle}}.$$

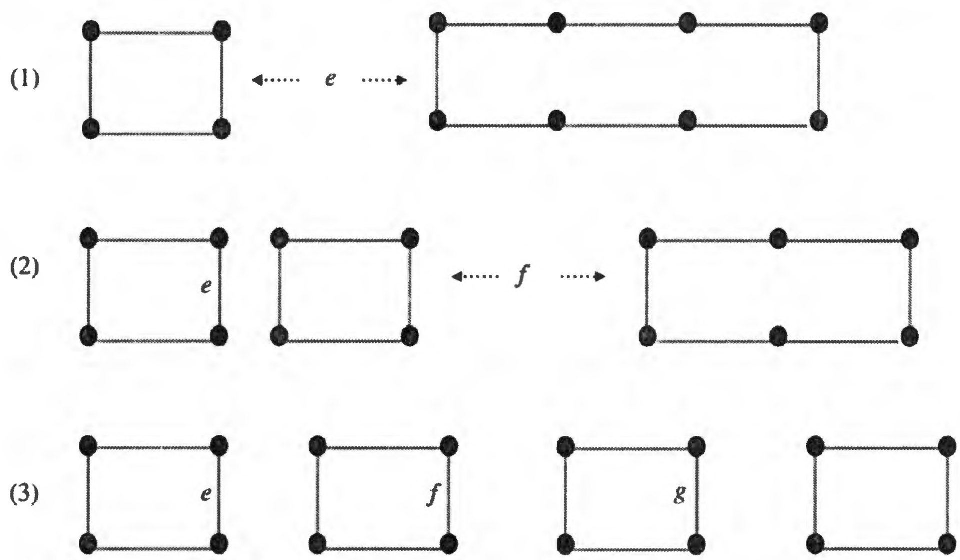



Figure 1.7: Factoring 4-cycles out of $G$ using flip chords

Notice $B_{\mathbf{8cycle}}$ corresponds to a cover of a square collection of size 8 obtained by removing the two left-most squares (i.e. vertices). Observe that $y^{T_1} - y^{T_2}$ is a product of cubics not quadratics. In order to demonstrate the desired result we continue the procedure until we reach the last 4-cycle as in Figure 1.3. We get $B_{T_1,T_2} =$

$$y_3y_5y_7(y_1y_9 - y_ey_{10}) + y_5y_{10}y_e(y_3y_7 - y_gy_f) + y_{10}y_ey_f(y_gy_5 - y_4y_6) + y_4y_6y_{10}(y_ey_f - y_2y_8).$$

and each of our flips corresponds to a 4-cycle.

## 1.4 Connectivity of Tilings of $2D$ Regions

We conclude this half of the thesis with showing any binomial arising from two tilings of a disjoint union $R$ of a set of contractible 2D cubiculated regions is generated by quadratics. This allows us to prove that the space of tilings of $R$ is flip-connected.

### 1.4.1 Results

**Theorem 1.35.** *Let R be the disjoint union of a set of contractible cubiculated regions. Any binomial arising from two distinct tilings of R is generated by quadratics. In particular,* $I_{R_{tiling}} \subseteq I_{R_{flip}}$.

*Proof.* We will proceed by induction on $k$, the size of the region $R$. For the base case, consider the case where $R$ is the $2 \times 2$ box, which is the smallest cubiculated region with at least two distinct tilings. In this case, $R$ has exactly two tilings, or equivalently, $G_R$ has two perfect matchings (these are the two matchings corresponding to the blue edges and red edges in Figure 1.8). Using the labeling from Figure 1.8, the binomial arising from the two corresponding tilings is $y_1y_3 - y_2y_4$, which is quadratic.

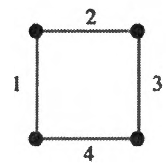



Figure 1.8: A 4-cycle with an associated binomial $y_1y_3 - y_2y_4$

Now assume the statement is true for all disjoint unions of contractible cubiculated regions of size $k-1$ or less and let $R$ be a region of size $k$. If $R$ is not tileable, then $I_{R_{tiling}}$ is the zero ideal, and the statement holds vacuously, thus we will assume that $R$ is tileable and that $T_1$ and $T_2$ are two possible coverings of $R$.

Let $B_{m,n}$ be an $m \times n$ box such that $R$ can be embedded into $B_{m,n}$. Embed $R$ in $B_{m,n}$. Let $i$ be the first row (working north to south) that contains a square of $R$ and $j$ be the first column of row $i$ (working left to right) that contains a square of $R$. We will call this square of $R$ embedded into the $(i,j)th$ square of $B_{m,n}$, the *leftmost corner of the top row* corner of $R$, and we will call the vertex of $G_R$ corresponding to this square, the *leftmost corner of the top row* of $G_R$.

### 1.4.2 Proof by Case Study Analysis

The proof will follow by cases, considering each of the ways $\mathcal{C} = T_1 \cup T_2$ can cover the top-most-western corner of $G_R$ (See Figure 1.9).

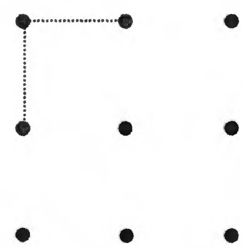


Figure 1.9: An uncovered corner.

Case 1: *The corner is covered by a 2-cycle (See Figure 1.10.)*

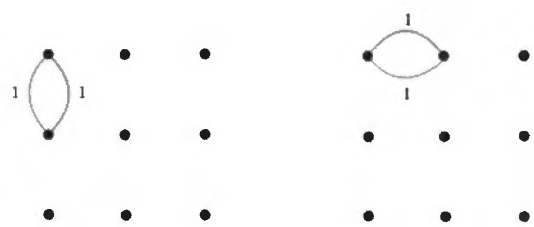



Figure 1.10: The case where the corner is covered by a 2-cycle

Our goal is to show $B_{T_1,T_2}$ is generated by quadratics. Let $T_1 = T_1 \setminus \{1\}$ and $T_2 = T_2 \setminus \{1\}$ be two tilings of the region $R \subseteq R$ obtained by removing the two squares corresponding to the vertices in the edge labeled 1 in Figure 1.10.

Then, we have

$$B_{T_1,T_2} = y_1 y^{T_1'} - y_1 y^{T_2'} = y_1(y^{T_1'} - y^{T_2'}).$$

Since $y^{T_1'} - y^{T_2'}$ is a binomial arising from two tilings of a cubiculated region $R'$ of size $k-2$ such that $R'$ is a disjoint union of contractible regions, we can apply our induction hypothesis. Thus, $y^{T_1'} - y^{T_2'}$ is generated by quadratics, and consequently, so is $B_{T_1,T_2}$.

Case 2: *The corner is covered by a 4-cycle as in Figure 1.11.*

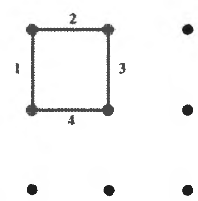



Figure 1.11: The case where the corner is covered by a 4-cycle.

In this case, we can obtain two smaller tilings

$$T_1' = T_1 \setminus \{1,3\} \qquad T_2' = T_2 \setminus \{2,4\},$$

such that $T_1'$ and $T_2'$ tile a subregion $R' \subseteq R$ obtained by removing the four squares corresponding to the vertices in the 4-cycle. By definition we get $B_{T_1,T_2} = y^{T_1} - y^{T_2}$. Notice we can write $B_{T_1,T_2}$ in the following form

$$y^{T_1} - y^{T_2} = y^{T_1'}(y_1y_3 - y_2y_4) + y_2y_4(y^{T_1'} - y^{T_2'}).$$

Since the binomial $y^{T_1'} - y^{T_2'}$ arises from two tilings of a contractible cubiculated region of size $k-4$, it is generated by quadratics and thus $B_{T_1,T_2}$ is generated by quadratics.

Case 3: *The corner of $G_R$ is covered by three edges of a square.*

In this case, there are four possible configurations that are illustrated in Figure 1.12 and Figure 1.13.

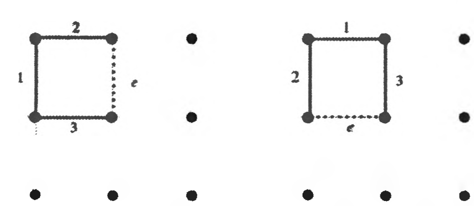



Figure 1.12: *Case 3.1* Two of the three edges belong to $T_1$.

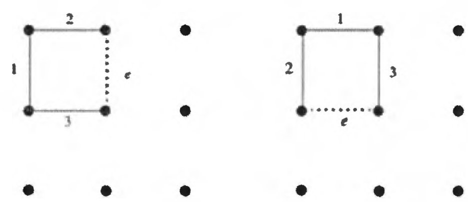



Figure 1.13: *Case 3.2* Two of the three edges belong $T_2$.

In all four of these configurations, the edge $e$ is a boundary flip chord of $T_1 \cup T_2$, thus we can apply Proposition 1.34 and our induction hypothesis to obtain the desired result.

Case 4: *The corner square of $G_R$ is covered by exactly two edges from $T_1 \cup T_2$. See Figure 1.14.*

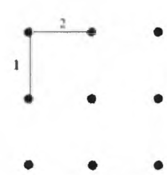



Figure 1.14: The corner is covered with exactly two edges from $T_1 \cup T_2$

In this case, we fix the north-west vertical edge to be from $T_1$. The arguments for the cases where north-west vertical edge is from $T_2$ are similar due to symmetry. Notice that since the corner square of $G_R$ is covered by exactly two edges of $T_1 \cup T_2$ and each vertex in $T_1 \cup T_2$ has degree two, then the north edge of the square to the right of the corner square is contained in $T_1 \cup T_2$. This is the most complicated case, and we will proceed by considering all the ways the south-east vertex of the corner square and its neighbor to the east can be covered.

(a) *The vertex immediately to the south-east of the north-west vertex is covered by a horizontal 2-cycle. This configuration is illustrated in Figure 1.15.*

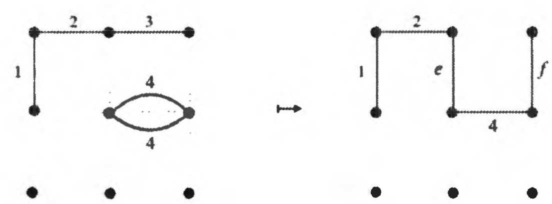



Figure 1.15: Case 4(a). The vertex immediately to the south-east of the north-west vertex is covered by a horizontal 2-cycle.

In this case, let $T_1' = T_1 \setminus \{3,4\} \cup \{e,f\}$. Then

$$B_{T_1,T_2} = y^{T_1 \setminus \{3,4\}}(y_3 y_4 - y_e y_f) + (y^{T_1'} - y^{T_2})$$

$$= y^{T_1 \setminus \{3,4\}}(y_3 y_4 - y_e y_f) + B_{T_1',T_2}.$$

Since $B_{T_1',T_2}$ is a binomial arising from two tilings of $R$ with a local corner configuration as in Case 3, it is generated by quadratics and thus $B_{T_1,T_2}$ is generated by quadratics.

(b) *The vertex immediately to the south-east of the north-west vertex is traversed with matching colors. This configuration is illustrated in Figure 1.16.*

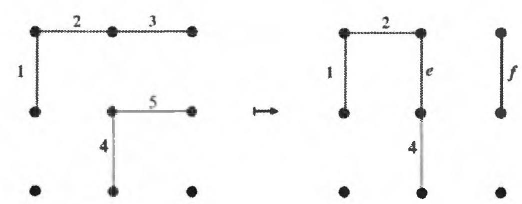



Figure 1.16: Case 4(b). The vertex immediately to the south-east of the north-west vertex is traversed with matching colors

In this case, let $T_1' = T_1 \setminus \{3,5\} \cup \{e,f\}$. Then

$$B_{T_1,T_2} = y^{T_1\setminus\{3,5\}}(y_3y_5 - y_ey_f) + (y^{T_1'} - y^{T_2})$$

$$= y^{T_1\setminus\{3,5\}}(y_3y_5 - y_ey_f) + B_{T_1',T_2}.$$

Since $B_{T_1',T_2}$ is a binomial arising from two tilings of $R$ with a local corner configuration as in Case 3, it is generated by quadratics and thus $B_{T_1,T_2}$

is generated by quadratics.

(c) *The vertex immediately to the south-east of the north-west vertex is traversed with non-matching colors. Illustrated in Figure 1.17.*

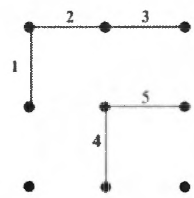



Figure 1.17: Case 4(c). The vertex immediately to the south-east of the north-west vertex is traversed with non-matching colors.

Let us denote the cycle containing $\{1,2,3\}$ as $\mathcal{C}_1$ and the cycle containing $\{4,5\}$ as $\mathcal{C}_2$. Note that $\{1,2,3\}$ and $\{4,5\}$ must be contained in disjoint cycles. To see this, assume the five edge are from the same cycle. Then either of the edges connecting edge 3 and edge 5 would be a chord. Since all chords of any cycle of $G_R$ are even, this would imply edge 3 and edge 5 are the same color, and thus we have a contradiction.

Next note that since $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint $\mathcal{C}_2$ is nested entirely inside $\mathcal{C}_1$. Let $R'$ be obtained by $R$ by deleting the squares corresponding to the vertices outside of $\mathcal{C}_2$ (viewing $G_R$ as the 1-skeleton of $R$ embedded in the

Euclidean plane), equivalently, $R'$ is the cubiculated region corresponding to $\mathcal{C}_2$ and its interior.

Let $T_1'$ be the subset of edges of $T_1$ that cover vertices of $G_{R'}$. Note that by construction $T_1'$ is a perfect matching of $G_{R'}$. Simarly, let $T_2'$ be the subset of edges of $T_2$ that cover vertices of $G_{R'}$. Then we have

$$B_{T_1,T_2} = y^{T_1 \setminus T_1'}(y^{T_1'} - y^{T_2'}) + (y^{T_1 \setminus T_1'} y^{T_2'} - y^{T_2}).$$

Since $R'$ is contractible with size less than $k$, the binomial $y^{T_1'} - y^{T_2'}$ is generated by quadratics. Furthermore, since $(T_1 \setminus T_1') \cup T_2'$ and $T_2$ are two tilings with a local corner configuration as Case 4(b), $y^{T_1 \setminus T_1'} y^{T_2'} - y^{T_2}$ is generated by quadratics. Therefore, $B_{T_1,T_2}$ is generated by quadratics.

*We now handle all the cases where the vertex immediately to the south-east of the north-west vertex is covered by a vertical 2-cycle.*

(d) *Two parallel vertical 2-cycles. This case is illustrated in Figure 1.18.*

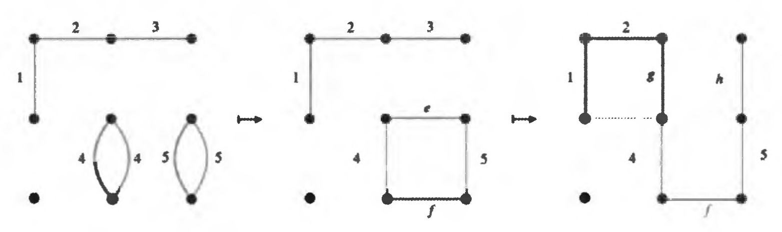



Figure 1.18: Case 4(d). Performing two flips on a corner configuration that contains two 2-cycles

Let

$$T_1' = T_1 \setminus \{3,4,5\} \cup \{f,g,h\} \qquad T_2' = T_2 \setminus \{e\}$$

Then,

$$B_{T_1,T_2} = y^{T_1\setminus\{4,5\}}(y_4y_5 - y_ey_f) + y^{T_1'\setminus\{g,h\}}(y_3y_e - y_gy_h) + (y^{T_1'} - y^{T_2'}).$$

Since the binomial $y^{T_1'} - y^{T_2'}$ is a binomial arising from two tilings of $R$ with a local corner configuration as in Case 3, it is generated by quadratics and thus $B_{T_1,T_2}$ is generated by quadratics.

(e) *A vertical 2-cycle and a traversed vertex with matching colors. Figure 1.19.*

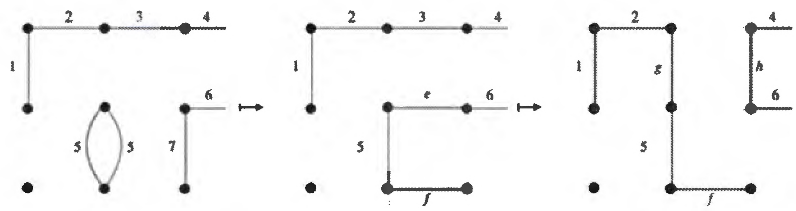



Figure 1.19: Case 4(e). A vertical 2-cycle and a traversed vertex with matching colors.

Let

$$T_1' = T_1 \setminus \{3,5,6\} \cup \{f,g,h\} \qquad T_2' = T_2.$$

Then

$$B_{T_1,T_2} = y^{T_1\setminus\{5,6\}}(y_5y_6 - y_ey_f) + y^{T_1\setminus\{3,5,6\}\cup\{f\}}(y_ey_3 - y_gy_h)$$
$$+(y^{T_1'} - y^{T_2'}).$$

Since $T_1'$ and $T_2'$ are tilings of $R$ with a local configuration as in Case 3, $y^{T_1'} - y^{T_2'}$ is generated by quadratics and thus so is $B_{T_1,T_2}$.

(f) *A vertical 2-cycle and a traversed vertex with opposite colors. Figure 1.20.*

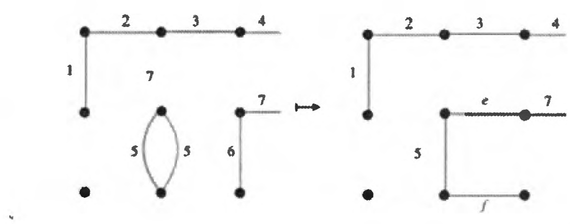



Figure 1.20: Case 4(f). A vertical 2-cycle and a traversed vertex with opposite colors.

Let

$$T_1' = T_1 \qquad T_2' = T_2 \setminus \{5,6\} \cup \{e,f\}.$$

Then,

$$B_{T_1,T_2} = y^{T_2 \setminus \{5,6\}}(y_5 y_6 - y_e y_f) + (y^{T_1'} - y^{T_2'}).$$

Notice $T_1'$ and $T_2'$ are tilings of $R$ that fit Case 4(c), so $y^{T_1'} - y^{T_2'}$ is generated by quadratics, and thus, so is $B_{T_1,T_2}$.

*Now we handle the cases where we have a vertical 2-cycle and a horizontal 2-cycle (See Figure 1.4.2). Since every vertex has degree* 2 *in* $G_R$ *then we know that the edge labeled 1 must be adjacent to blue edge going south or west, which splits these cases into two natural groups. We start with the case where the south edge is chosen.*

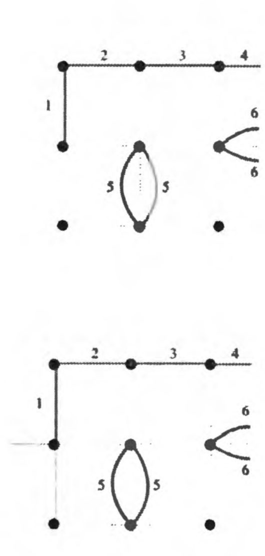



Figure 1.21: A vertical 2-cycle and a horizontal 2-cycle covering the internal vertices.

(g) *South edge is chosen. See Figure 1.22*

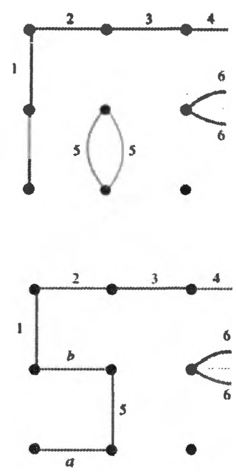



Figure 1.22: Case 4(g). Two adjacent north-south edges extending from corner.

In this case, we will perform a flip as illustrated in Figure 1.22. Let

$$T_1' = T_1 \qquad T_2' = T_2 \setminus \{a, 5\} \cup \{b, c\}.$$

Then

$$B_{T_1,T_2} = y^{T_2 \setminus \{a,5\}}(y_a y_5 - y_b y_c) + (y^{T_1'} - y^{T_2'}).$$

Since $B_{T_1',T_2'} = y^{T_1'} - y^{T_2'}$ is generated by quadratics as shown in Case 3, we have that $B_{T_1,T_2}$ is generated by quadratics.

*Now we will consider the cases where the edge to the west, labeled a in Figure 1.23, is chosen.*

*In order to explore this case we must expand our view of $G_R$ by considering the next column of vertices. We know that the edge labeled 4 is parallel to a 2-cycle and every vertex in $T_1 \cup T_2$ has degree 2. Thus, there exists an edge $b$ to the right of 4 that will be colored red. The cases will be enumerated based on the possible coverings of the highlighted vertex in Figure 1.23.*

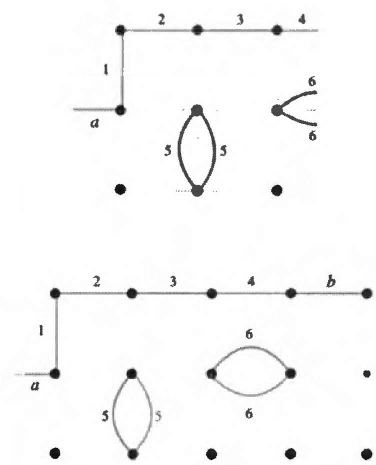



Figure 1.23: Expanding our view of $T_1 \cup T_2$.

*Since we have seen how flip moves correspond to quadratics, for most of these cases we will show how a sequence of flips can transform $T_1 \cup T_2$ to a configuration in a previous case or to a configuration with a boundary flip chord.*

(h) *Edge extending south from edge $b$. See Figure 1.24.*

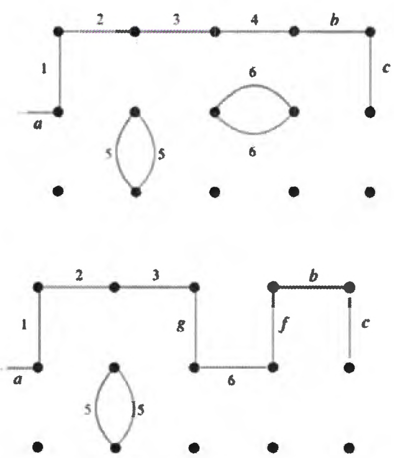



Figure 1.24: Case 4(h). Edge extending south from edge $b$.

Performing a single flip as illustrated in Figure 1.24 moves us to configuration with a boundary flip chord, and we can apply Proposition 1.34. *Now we may assume that there is a blue horizontal edge* $\{h\}$*. We still need to consider several subcases where the highlighted vertex in Figure 1.23 is covered by a vertical* 2*-cycle which we label* $\{d\}$*.*

(i) *The vertices south and parallel to the edge labeled 6 are covered by two vertical* 2*-cycles. See Figure 1.25.*

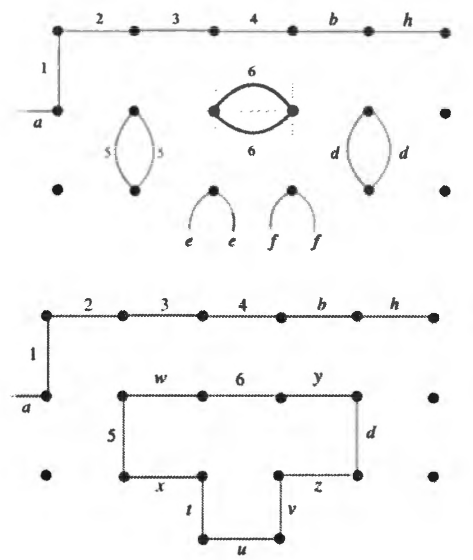



Figure 1.25: Case 4(i). The vertices south and parallel to the edge labeled 6 are covered by two vertical 2-cycles.

Notice that we have a collection of five 2-cycles, we first will move the red edges to form an interior 10-cycle. Let

$$T_1' = \{e, f, d, 5, 6\} \qquad T_2' = \{u, w, x, y, z\}$$

and let

$$T_1'' = (T_1 \setminus T_1') \cup T_2' \qquad T_2'' = T_2.$$

Then, we have

$$B_{T_1,T_2} = y^{T_1 \setminus T_1'}(y^{T_1'} - y^{T_2'}) + (y^{T_1''} - y^{T_2''}).$$

Since $T_1'$ and $T_2'$ tile a contractible cubiculated region $R'$ of size less than $k$, the binomial $(y^{T_1'} - y^{T_2'})$ is generated by quadratics. Furthermore, $y^{T_1''} - y^{T_2''}$ is generated by quadratics by Case 4.3. Therefore, $B_{T_1,T_2}$ is generated by quadratics.

(j) *The edge south of* $\{6\}$, *belongs to a horizontal* 2*-cycle. See Figure 1.26.*

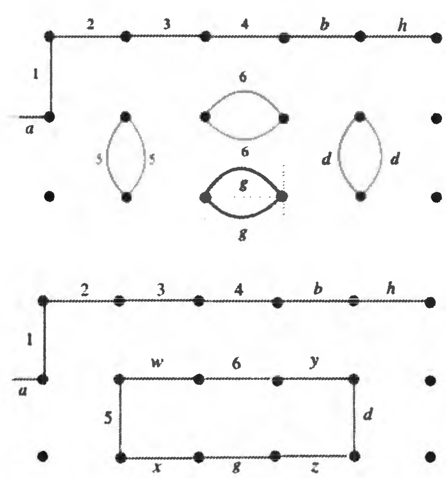



Figure 1.26: Case 4(j). The edge south of $\{6\}$, belongs to a horizontal 2-cycle.

In this case, we move the red edges of the four 2-cycles to obtain an 8-cycle, and the proof follows similarly to Case 4(i).

(k) *An edge south of* $\{6\}$ *that is colored blue and not part of a 2-cycle. See Figure 1.27.*

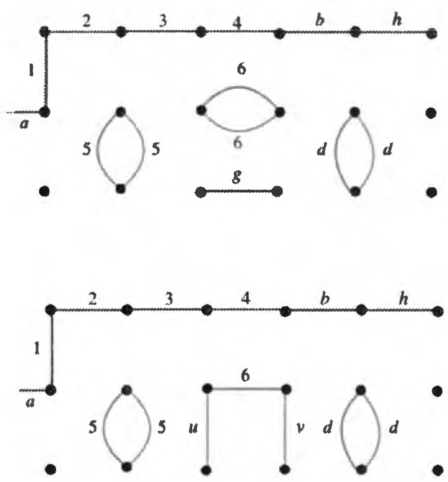



Figure 1.27: Case 4(k). An edge south of $\{6\}$ that is colored blue and not part of a 2-cycle.

Flipping the edges $\{6, g\}$ to $\{u, v\}$ as in Figure 1.27 gives us a configuration as in Case 4(f).

(l) *An edge south of* $\{6\}$ *that is colored red and not part of a 2-cycle. See Figure 1.28.*

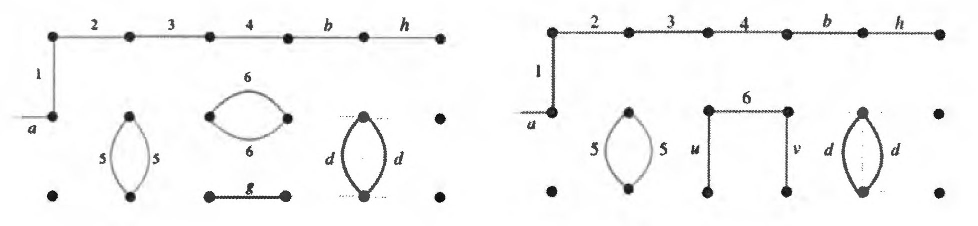



Figure 1.28: Case 4(l). An edge south of $\{6\}$ that is colored red and not part of a 2-cycle.

Flipping the red edges $\{6, g\}$ to $\{u, v\}$ as in Figure 1.27 gives us a configuration as in Case 4(e). *This concludes the cases where the marked vertex is covered by a* 2*-cycle.*

(m) *Vertex is covered by a horizontal* 2*-cycle. See Figure 1.29.*

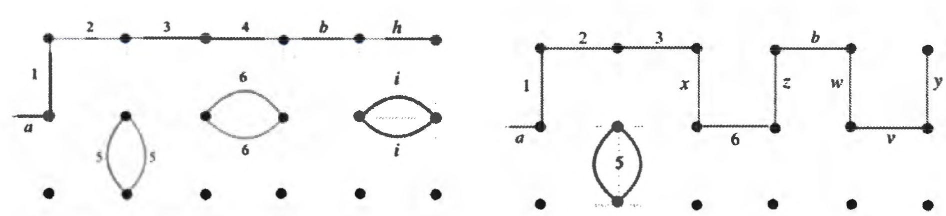



Figure 1.29: Case 4(m). Vertex is covered by a horizontal 2-cycle.

Flipping $\{4,6\}$ to $\{x,z\}$ and $\{h,i\}$ to $\{w,y\}$ as in Figure 1.29 results in a configuration with a boundary flip chord.

(n) *Vertex is covered a horizontal blue edge and a vertical edge red. See Figure 1.30.*

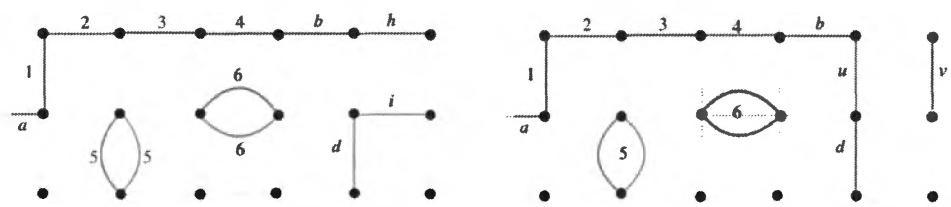



Figure 1.30: Case 4(n). Vertex is covered a horizontal blue edge and a vertical edge red.

Let $\{u,v\}$ be the edges replacing $\{h,i\}$ after performing a flip. The configuration has the form of Case 4(h).

(o) Vertex is covered a vertical blue edge and a horizontal red edge. See Figure 1.31.

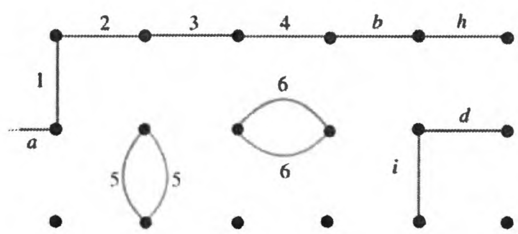



Figure 1.31: Case 4(o). Vertex is covered a vertical blue edge and a horizontal red edge.

This is handled using an argument similar to case 4(c).

By an exhaustive case study, we have shown $B_{T_1,T_2}$ is generated by quadratics for all possible corner configurations of $T_1 \cup T_2$. $\square$

### 1.4.3 Implying Thurston

We can see that Theorem 1.35 gives us the following corollary,

**Corollary 1.36.** *Any two tilings of a contractible proper 2D homogeneous cubical complex are flip-connected.*

*Proof.* This follows by combining Corollary 1.26 and Theorem 1.35. $\square$

# Chapter 2

# Polytopes and Matroids

## 2.1 Background

### 2.1.1 Polytopes

Some of the nicest and most well understood polytopes around are convex and simplicial.

**Definition 2.1.** A *$d$-dimensional convex polytope* $P$ is the convex hull of finitely many points in $\mathbb{R}^d$. Equivalently, $P$ can be described as the bounded intersection of finitely many hyperplanes.

**Definition 2.2.** A *simplicial $d$-dimensional polytope* $P$ for $d \geq 3$ dimensions contains only triangular faces (*simplices*).

**Definition 2.3.** A *$d$-dimensional face* of $P$ is the intersection $P \cap H$, where $H$ is a supporting hyperplane (a hyperplane such that $P$ lies completely to one side of $H$).

**Definition 2.4.** The *$f$-vector* $f(P) = (f_0, f_1, \ldots, f_d)$ is a vector such that $f_i$ equals the number of $i$-dimensional faces of $P$.

One can further encode this information as the *$h$-vector* $h(P) = (h_0, h_1, \ldots, h_d)$ via the relation

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

Note, this relation holds because $P$ is simplicial and convex.

It was conjectured by McMullen [14] and subsequently proven by Billera–Lee, and Stanley [4], [5], and [20] that $f$-vectors of simplicial polytopes can be fully characterized by the $g$-theorem. The $g$-theorem determines whether a vector of positive integers is indeed the $f$-vector of some simplicial polytope.

### 2.1.2 Sufficient Subsets of the $f$-vector

The Dehn–Sommerville relations condense the $f$-vector into the $g$-vector, which has length $\lceil \frac{d+1}{2} \rceil$. This raises the question: Which $(\lceil \frac{d+1}{2} \rceil)$-subsets of the $f$-vector of a general simplicial polytope are sufficient to determine the whole $f$-vector?

**Definition 2.5.** Define a *Dehn–Sommerville basis* to be a minimal subset $S$ such that $\{f_i \mid i \in S\}$ determines the entire $f$-vector for any simplicial polytope.

**Example 2.1.** Using the $g$-theorem one can check that $f(P_1) = (1, 8, 27, 38, 19)$ and $f(P_2) = (1, 9, 28, 38, 19)$ are $f$-vectors of two different simplicial 4–polytopes.

Therefore the entries $f(P) = (1, *, *, 38, 19)$ do not determine a simplicial $f$-vector uniquely, and $\{1, 4, 5\}$ is not a Dehn-Sommerville basis in dimension 4.

The following work was done in collaboration with Anastasia Chavez. In this paper we prove the following theorem.

**Theorem 2.6.** *[?] The Dehn–Sommerville bases of dimension $2n$ are precisely the upstep sets of the Dyck paths of length $2(n+1)$.*

A similar description holds for the Dehn-Sommerville bases of dimension $2n-1$.

### 2.1.3 Matroids

A matroid is a combinatorial object that generalizes the notion of independence. We provide the definition in terms of bases; there are several other equivalent axiomatic definitions available. For a more complete study of matroids see Oxley [17].

**Definition 2.7.** A *matroid* $M$ is a pair $(E, \mathcal{B})$ consisting of a finite set $E$ and a nonempty collection of subsets $\mathcal{B} = \mathcal{B}(M)$ of $E$, called the *bases* of $M$, that satisfy the following properties:

(B1) $\mathcal{B} \neq \emptyset$

(B2) (Basis exchange axiom) If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists an element $b_2 \in B_2 - B_1$ such that $B_1 - \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

**Example 2.2.** A key example is the matroid $M(A)$ of a matrix $A$. Let $A$ be a $d \times n$ matrix of rank $d$ over a field $K$. Denote the columns of $A$ by $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n} \in K^d$. Then $B \subset [n]$ is a basis of $M(A)$ on the ground set $[n]$ if $\{\mathbf{a_i} \mid i \in B\}$ forms a linear basis for $K^d$.

### 2.1.4 The Dehn-Sommerville relations

**Definition 2.8.** Let $P$ be a $d$-dimensional simplicial polytope, that is, a polytope whose facets are simplices. Then define $f(P) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$ to be the *$f$-vector of $P$* where $f_i$ is the number of $i$-dimensional faces of $P$. It is convention that $f_{-1} = 1$.

**Definition 2.9.** For $k \in [0, d]$, the *$h$-vector* of a simplicial polytope is a sequence with elements

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$

**Definition 2.10.** The *$g$-vector* of a simplicial polytope is a sequence where $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $i \in [1, \lfloor \frac{d}{2} \rfloor]$.

The Dehn–Sommerville relations can be stated most simply in terms of the $h$-vector.

**Theorem 2.11.** *[13] The h-vector of a simplicial d-polytope satisfies*

$$h_k = h_{d-k}$$

*for* $k = 0, 1, \ldots, d$.

We now discuss a matrix reformation of Theorem

**Definition 2.12.** The *Dehn-Sommerville matrix*, $M_d$, is defined by

$$(M_d)_{\substack{0 \le i \le \lfloor \frac{d}{2} \rfloor \\ 0 \le j \le d}} := \binom{d+1-i}{d+1-j} - \binom{i}{d+1-j},$$

for $d \in \mathbb{N}$.

**Theorem 2.13.** *[6] Let* $P$ *be a simplicial d-polytope, and let* $f$ *and* $g$ *denote its* $f$ *and g-vectors. Then*

$$g \cdot M_d = f.$$

Label the columns of $M_d$ from 1 to $d+1$. Just as in the previous example, we can define the *Dehn-Sommerville matroid* of rank $d$, $DS_d$, to be the pair $([d+1], \mathcal{B})$ where $B \in \mathcal{B}$ if $B$ is a collection of columns associated with a non-zero maximal minor.

**Example 2.3.** Let $d = 4$. Then

$$M_4 = \begin{pmatrix} 1 & 5 & 10 & 10 & 5 \\ 0 & 1 & 4 & 6 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

and the bases of $DS_4$ are

$$\mathcal{B} = \{123, 124, 125, 134, 135\}$$

where basis $ijk$ refers to the submatrix formed by columns $i, j$, and $k$ in $M_4$.

**Definition 2.14.** Define the graph $\mathbf{DS}_d$ to be a directed graph (or *digraph*), as shown in Figure where all horizontal edges are directed east and all vertical edges are directed north.

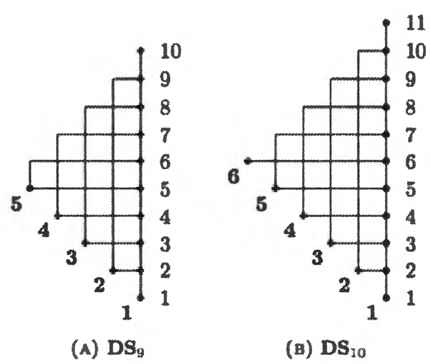



Figure 2.1: The Dehn–Sommerville graphs for odd and even values of $d$.

**Definition 2.15.** Number the nodes along the southwest border $1, \ldots, \lceil \frac{d+1}{2} \rceil$ in bold, and call them *sources*. Number the nodes along the east border $1, \ldots, d+1$, and call them *sinks*. A *routing* is a set of vertex-disjoint paths on a digraph.

Considering the collection of routings on $\mathbf{DS}_d$ produces the following theorem.

**Theorem 2.16.** *A subset $B \subset [d+1]$ is a basis of $DS_d$ if and only if there is a routing from the source set $[\lceil \frac{\mathbf{d+1}}{\mathbf{2}} \rceil]$ to the sink set $B$ in the graph $\mathbf{DS}_d$.*

*Proof.* Björklund and Engström [3] found a way to give positive weights to the edges of $\mathbf{DS}_d$ so that $M_d$ is the path matrix of $\mathbf{DS}_d$; that is, entry $(M_d)_{ij}$ equals the sum of the product of the weights of the paths from source $\mathbf{i}$ to sink $j$. Then, by the Lindström–Gessel–Viennot lemma CITE, the determinant of columns $\mathbf{a}_{\mathbf{i}_1}, \ldots, \mathbf{a}_{\mathbf{i}_{\lceil \frac{d+1}{2} \rceil}}$ is the sum of the product of weights of the routing from the source nodes $\mathbf{i}_1 < \cdots < \mathbf{i}_{\lceil \frac{d+1}{2} \rceil}$ to the sink nodes $a_{i_1}, \ldots, a_{i_{\lceil \frac{d+1}{2} \rceil}}$. It follows that the determinant is non-zero if and only if there is a routing. □

**Example 2.4.** Continuing with $d = 4$, we see the destination sets of the routings on digraph $\mathbf{DS}_4$ match with the bases of $DS_4$ found in Example.

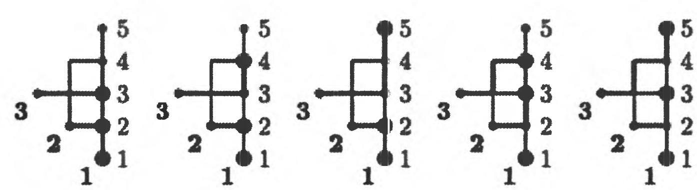



Figure 2.2: The bases for $d = 4$

### 2.1.5 The Catalan Matroid

**Definition 2.17.** For $n \in \mathbb{N}$, a *Dyck path* of length $2n$ is a path in the plane from $(0,0)$ to $(2n,0)$ with upsteps, $(1,1)$, and downsteps, $(1,-1)$, that never falls below the $x$-axis. The number of Dyck paths of length $2n$ is counted by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

**Definition 2.18.** [1, 8] The *Catalan matroid*, $\mathbf{C}_n$, is a matroid with a ground set of $[2n]$ whose bases are the upstep sets of the Dyck paths of length $2n$.

**Example 2.5.** Let $n = 3$. Then the bases of the Catalan matroid are the upstep sets of the following Dyck paths of length 6:

$$\mathcal{B} = \{123, 124, 125, 134, 135\}$$

Figure 2.3: All Dyck paths of length 6.

## 2.2 Result

[?] The Dehn–Sommerville matroids are obtained from the Catalan matroids by removing trivial elements:

$$DS_{2n} \cong \mathbf{C}_{n+1}\backslash(2n+2) \quad \text{and} \quad DS_{2n-1} \cong \mathbf{C}_{n+1}\backslash 1\backslash(2n+2) \quad \text{for } n \in \mathbb{N}.$$

Note that 1 is a coloop and $2n+2$ is a loop in $\mathbf{C}_{n+1}$.

To prove our main result, we first set the stage with some preliminary lemmas. We first show there is a close connection between odd and even Dehn–Sommerville matroids. We then characterize the bases of the Dehn–Sommerville and Catalan matroids, respectively. Finally, we prove our main result.

**Lemma 2.19.** *The bases of the Dehn–Sommerville matroid $DS_{2n}$ are in bijection with the bases of the Dehn–Sommerville matroid $DS_{2n-1}$. More precisely, $DS_{2n} \cong DS_{2n-1} \oplus C$, where $C$ is a coloop.*

*Proof.* For the graph $\mathbf{DS}_{2n}$, any path leaving the top source node $\lceil \mathbf{\frac{2n+1}{2}} \rceil = n+1$ must first travel east along the only edge leaving it. To satisfy the condition that all paths in a routing are vertex-disjoint, any path leaving all other source nodes, excluding **1**, must also first travel east. Source node **1** is forced to have the trivial path. Thus, the horizontal edges from every source node, excluding **1**, can be contracted without affecting the potential destinations of the routings. This results in a graph equal to the graph $\mathbf{DS}_{2n-1}$ with an added vertical edge extending from the bottom.

It follows that the number of bases of $DS_{2n}$ is equal to the number of bases of $DS_{2n-1}$ and there exists a bijection between the bases; namely,

$$\{1, b_2, b_3, \ldots, b_n\} \mapsto \{b_2 - 1, b_3 - 1, \ldots, b_n - 1\},$$

where $\{1, b_2, b_3, \ldots, b_n\}$ is a basis of $DS_{2n}$. □

**Lemma 2.20.** *[21] Let $a_1 < a_2 < \cdots < a_n$ be the upsteps of a lattice path $P$ of length $2n$. Then $P$ is a Dyck path if and only if*

$$a_1 = 1, a_2 \leq 3, a_4 \leq 5, \ldots, a_n \leq 2n - 1.$$

**Lemma 2.21.** *Let $B = \{b_1 < b_2 < \cdots < b_{n+1}\}$ be a subset of $[2n + 1]$. Then $B$ is a basis of $DS_{2n}$ if and only if $b_1 = 1, b_2 \leq 3, b_3 \leq 5, \ldots, b_{n+1} \leq 2n + 1$.*

*Proof.* First let us prove the forward direction. Assume $B = \{b_1 < b_2 < \cdots < b_{n+1}\}$ is a basis of $DS_{2n}$. Assume $b_i \geq 2i$. Notice that the northwest diagonal starting at sink $2i$ has $n - i$ vertices (excluding sink $2i$). If there was a routing, the $n + 1 - i$ paths starting at source nodes $\mathbf{i+1}, \ldots, \mathbf{n+1}$ would pass through this northwest diagonal, which forms a bottleneck of width $n - i$, contradicting the presence of a routing. Thus, assuming a routing exists from sources [i] to sinks $b_1, \ldots, b_i$, the remaining source nodes $\mathbf{i+1}, \ldots, \mathbf{n+1}$ can only be routed if $b_i \leq 2i - 1$. See the below image for an example of a sink set that creates a bottleneck when $2n = 10$

and $b_4 = 8$.

To prove the other direction, assume $b_1 = 1, b_2 \leq 3, b_3 \leq 5, \ldots, b_{n+1} \leq 2n+1$. Construct a routing sequentially from the bottom, making each path as low as possible. All routings must begin with $b_1 = 1$. For each $i$, we need to check whether the number of nodes remaining on the northwest diagonal containing $b_i$ is greater than or equal to the number of remaining sink nodes. Since $b_i \leq 2i-1$, then by our previous argument no bottlenecks are formed and this gives us a valid routing, so $B = \{b_1 < b_2 < \cdots < b_{n+1}\}$ is a basis of $DS_{2n}$. □

### 2.2.1 Proof of Theorem

*Proof of Theorem 2.2.* This follows immediately from Lemmas 2.19, 2.20, and 2.21. □

Note that Theorem 2.6 describes in words the bases of the matroids involved in Theorem 2.2.

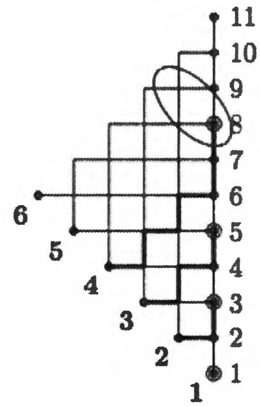



Figure 2.4: Bottleneck prevents a routing for sinks $(1, 3, 5, 8, *, *)$ for $d = 2n = 10$

**Remark.** Björner [7] conjectured and Björklund and Engstöm [3] proved that $M_d$ is totally non-negative, which implies $DS_d$ is a positroid. A corollary to our main theorem is that the Catalan matroid is a positroid, a result recently discovered by Pawlowski [18] via pattern avoiding permutations. Thus, there are several positroid representations introduced by Postnikov [19] that can be used to further represent the Catalan matroid.

Using Lemma 2.21, one can verify that the decorated permutation corresponding to the positroid $\mathbf{C}_{n+1}$ is

$$
\begin{array}{ll}
\bar{1}35\cdots(2n+1)246\cdots(2n) & \text{when} \;\; DS_{2n} \cong \mathbf{C}_{n+1}\backslash(2n+2) \\
246\cdots(2n)135\cdots(2n-1) & \text{when} \;\; DS_{2n-1} \cong \mathbf{C}_{n+1}\backslash 1\backslash(2n+2)
\end{array}
$$

for $n \in \mathbb{N}$.

We can use this relation to obtain the Grassmann necklace, Le diagram and juggling pattern corresponding to $\mathbf{C}_{n+1}$.

# Bibliography


[1] F. Ardila, *The Catalan Matroid*, J Comb Theory A **104** (2003), 49–62.

[2] R. Berger, *The Undecidability of the Domino Problem*, Memoirs of the American Mathematical Society **66** (1966).

[3] M. Björklund and A. Engstöm, *The g-Theorem Matrices are Totally Nonnegative*, J. Combin. Theory Ser. A **116** (2009), 730–732.

[4] A. Björner, *The Unimodality Conjecture for Convex Polytopes*, Bull. Amer. Math. Soc. **4** (1981), 187–188.

[5] A. Björner, *Face Numbers of Complexes and Polytopes*, Proc. Intern. Congr. Math., Berkeley, CA **1408-1418** (1986).

[6] A. Björner, *Partial Unimodality for $f$-vectors of Simplicial Polytopes and Spheres*, "Jerusalem Comb. '93" (H. Barcelo and G. Kalai, eds.), Contemp. Math. **178** (1994), 45–54, *Amer. Math. Soc.*

[7] A. Björner, *Theorem for $f$-vectors of Simplicial Polytopes*, Pure Appl. Math. Q **3 (1)** (2007), 347–356.

[8] J. Bonin, A. de Mier, and M. Noy, *Lattice Path Matroids: Enumerative Aspects and Tutte Polynomials*, J. Combin. Theory Ser. A **104** (2003), 63–94.

[9] D. Dummit and R. Foote, *Abstract algebra*, 1991.

[10] M. Fisher and H. Temperley, *The Dimer Problem in Statistical Mechanics–an Exact Result*, Philosophical Magazine (1961), 1061–1063.

[11] J. Freire, C. Klivans, P. Milet, and N. Saldanha, *On the Connectivity of Spaces of Three-dimensional Domino Tilings*, (2017).

[12] P. Kasteleyn, *The Statistics of Dimers on a Lattice I. The Number of Dimer Arrangements on a Quadratic Lattice*, Physica **27** (1961), 1209–1225.

[13] P. McMullen, *The Maximum Number of Faces of a Convex Polytope*, Mathematika **17** (1970), 179–184.

[14] P. McMullen, *The Number of Faces of a Simplicial Polytope*, Israel Journal of Mathematics **9** (1971), 559–570.

[15] H. Ohsugi and T. Hibi, *Toric Ideals Generated by Quadratic Binomials*, Journal of Algebra **218** (1999), 509–527.

[16] H. Ohsugi and T. Hibi, *Toric Ideals and Their Circuits*, Journal of Commutative Algebra **5** (2012), 309–322.

[17] J. Oxley, *Matroid Theory*, 2nd ed., Oxford University Press Inc., 1992.

[18] B. Pawlowski, *Catalan Matroid Decompositions of Certain Positroids*, arXiv:1502.00158 [math.CO].

[19] A. Postnikov, *Total Positivity, Grassmannians, and Networks*, Preprint. Available at http://www-math.mit.edu/~apost/papers/tpgrass.pdf.

[20] R. Stanley, *The Upper Bound Conjecture and Cohen-Macaulay Rings*, Stud. Appl. Math. **54** (1975), 135–142.

[21] R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, 1999.

[22] W. Thurston, *Three-Dimensional Geometry and Topology of Three-Manifolds*, 1st ed., Princeton University Press, 1997.

[23] R. Villareal, *Monomial Algebras*, 2nd ed., CRC Press, 2015.

[24] H. Wang, *Proving Theorems by Pattern Recognitionii*, Bell System Technical Journal **40** (1961), 1–41.

[25] D. West, *Introduction to Graph Theory*, 2nd ed., Pearson, 2015.