EXACTNESS AND OVERCOMPLETNESS OF FOURIER FRAMES FOR FRACTAL MEASURES

A thesis submitted to the faculty of
San Francisco State University
in partial fulfillment of
the requirements for
the degree of

Master of Arts
In
Mathematics

by

Shahram Emami
San Francisco, California

September 2018

CERTIFICATION OF APPROVAL

I certify that I have read Exactness and Overcompletness of Fourier Frames for Fractal Measures by Shahram Emami and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillllment of the requirements for the degree of Master of Arts in Mathematics at San Francisco State University.

Emily Clader

Assistant Professor of Mathematics

Alexander Schuster

Professor of Mathematics

Chun-Kit Lai

Assistant Professor of Mathematics

ACKNOWLEDGMENTS

I would like to express my heartfelt thanks to Dr. Chun Kit Lai for being patient with me while I was finding my thesis topic and for teaching me real, complex and functional analysis. I would like to thank Dr. Emily Clader for being a member of my thesis committee, even though my thesis topic is not directly related to algebraic geometry and for her clear explanation of concepts in algebra and topology. I would like to also thank Dr. Alex Schuster for being a committee member and for his teachings in advanced linear algebra and undergraduate real analysis and topology which provided a foundation for my success as a mathematics graduate student.

EXACTNESS AND OVERCOMPLETNESS OF FOURIER FRAMES FOR FRACTAL MEASURES

Shahram Emami

San Francisco, California

2018

In 1952 Duffin and Schaeffer introduced the idea of frames in a Hilbert space [1]. Whereas a basis as studied by students in introductory linear algebra class is a spanning set that is also linearly independent, frames have to span the space but they need not be linearly independent. The relaxation of the linear independence requirement has made frames applicable to a wide variety of problems in applied harmonic analysis and signal processing including the problem of data transmission in a noisy channel. A set $\Omega \subset \mathbb{R}^d$ with positive finite Lebesgue measure is said to be a spectral set if there exists a $\Lambda \subset \mathbb{R}^d$ such that $\{e^{2\pi i\langle\lambda,\cdot\rangle}\}_{\lambda\in\Lambda}$ is an orthogonal basis of $L^2(\Omega)$. The set Λ is then said to be a spectrum of Ω and (Ω, Λ) is called a spectral pair. A challenging problem in frame theory is construction of Fourier frames or exponential orthonormal bases in different measures spaces $L^2(\mu)$, particularly when μ is a so called "fractal measure" (i.e. when the support is a fractal set). It has been shown that the standard one-third Cantor measure is not a spectral measure, while the standard one-fourth measure is [2]. Previously almost-Parseval frame tower (APFT) was given as a pathway to construct Fourier frames on convolutional-type fractal measures [3]. Here, we extend the previous results by: (1) Giving necessary and sufficient conditions for the frame constructed by APFT to be exact. (2) Finding a measure which admits a Fourier frame with ∞-redundancy.

I certify that the abstract is a correct representation of the content of this thesis.

Chun-Kit Lai 12-11-2018

Chair, Thesis Committee Date

Table of Contents

Chapter 1

Frame Basics

Definition 1.1. A countable family of vectors $\{\varphi_i\}_{i\in I}$ is called a frame for the Hilbert space $\mathcal{H}$ if there exist constants $0 < A \leq B < +\infty$ such that

$$A\,||x||^2 \leq \sum_{i\in I} |\langle x, \varphi_i\rangle|^2 \leq B\,||x||^2. \tag{1.1}$$

If $A = B$, then we have a tight frame. If $A = B = 1$, then we have a Parseval frame. An exact frame is one that ceases to be a frame whenever any single element is deleted from its sequence. A frame which is not a basis is said to be *overcomplete* or *redundant* [4]. Let $\{e_n\}$ be the standard orthonormal basis for a Hilbert space $\mathcal{H}$. Then:

- $\{e_n\}$ is a tight exact frame for $\mathcal{H}$ with frame bounds $A = B = 1$ (Parseval frame).
- $\{e_1, e_1, e_2, e_2, e_3, e_3, \cdots\}$ is a tight inexact frame with bounds $A = B = 2$, but it is not orthogonal and therefore not a basis.
- $\{\frac{e_1}{1}, \frac{e_2}{2}, \frac{e_3}{3}, \cdots\}$ is a complete orthogonal sequence and it is a basis for $\mathcal{H}$, but it is not a frame since the lower bound condition is not met.

- $\{2e_1, e_2, , e_3, \cdots\}$ is a non-tight exact frame with bounds $A = 1, B = 2$.

1.1 Frame operators and their properties

Given a finite or infinite index set I, the square summable sequences are defined as: $\ell^2(I) = \{(a_n)_{n\in I} : \sum_{n\in I} |a_n|^2 < +\infty\}$. Let $\{\varphi_i\}_{i\in I}$ be a frame. Then the associated *analysis* operator T is defined as:

$$T : \mathcal{H} \to \ell^2(I)$$
$$Tx = (\langle x, \varphi_i \rangle)_{i\in I}$$

For any $z \in \mathbb{C}$, $|z|^2 = z\overline{z}$ where $\overline{z}$ is the complex conjugate of z, hence $\|Tx\|^2 = \langle Tx, Tx \rangle = \sum_{i\in I} |\langle x, \varphi_i \rangle|^2$. If $\langle Tx, y \rangle_{\ell^2} = \langle x, T^*y \rangle_{\mathcal{H}}$, $\forall x \in \mathcal{H}, \forall y \in \ell^2(I)$, then $T^* : \ell^2 \to \mathcal{H}$ is the adjoint of T and is known as the *synthesis* operator.

$$\begin{aligned}\langle Tx, y \rangle &= \langle ((\langle x, \varphi_i \rangle), (y_i) \rangle_{i\in I} \\ &= \sum_{i\in I} \langle x, \varphi_i \rangle \, \overline{y_i} \\ &= \langle x, \sum_{i\in I} y_i \, \varphi_i \rangle.\end{aligned}$$

Therefore $T^*(y_i)_{i\in I} = \sum_{i\in I} y_i \, \varphi_i$. Given a frame $\{\varphi_i\}_{i\in I}$ on $\mathcal{H}$, the frame operator S is defined as:

$$S : \mathcal{H} \rightarrow \mathcal{H}$$
$$S = T^*T$$
$$Sx = \sum_{i=1}^{M} \langle x, \varphi_i \rangle \varphi_i.$$

Notice that $\langle Sx, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \sum_{i\in I} |\langle x, \varphi_i\rangle|^2$, and as a result the frame inequality is equivalent to $A\ ||x||^2 \leq \langle Sx, x\rangle \leq B\ ||x||^2$. An operator T is called positive if $\langle Tx, x\rangle \geq 0$, $\forall x \in \mathcal{H}$. Thus $T \geq S \Leftrightarrow (T - S)$ is positive. For the frame operator this implies $\langle Ax, x\rangle \leq \langle Sx, x\rangle \leq \langle Bx, x\rangle$, $\forall x \in \mathcal{H}$ and hence $AI \leq S \leq BI$, where I is the N-dimensional identity operator. For a tight frame this implies:

$$\sum_{i\in I} |\langle x, \varphi_i\rangle|^2 = A\ ||x||^2 = \langle Sx, x\rangle \Rightarrow S = AI \Rightarrow S^{-1} = \frac{1}{A} I$$

Theorem 1.1 (*Fundamental Theorem of Frames*). *The frame operator S is positive, self-adjoint and invertible.*

Proof. $0 \leq \langle Ax, x\rangle \leq \langle Sx, x\rangle$, $\forall x \in \mathcal{H}$ and hence S is positive. $S^* = (T^*T)^* = T^*T = S$ which proves S is self-adjoint. To show that S is invertible, notice that:

$$
\begin{aligned}
AI &\leq S \leq BI \\
AI - BI &\leq S - BI \leq 0 \\
BI - AI &\geq BI - S \geq 0 \\
\frac{B-A}{B} I &\geq I - B^{-1}S \geq 0 \\
||I - B^{-1}S|| = \sup_{||f||=1} |\langle (I - B^{-1}S)f, f\rangle| &\leq \frac{B-A}{B} < 1
\end{aligned}
$$

which implies that S is invertible by Lemma 4.1 completing the proof.

□

Theorem 1.2. *Let* $\Phi = \begin{bmatrix} | & & | \\ \varphi_1 & \dots & \varphi_n \\ | & & | \end{bmatrix} = \begin{bmatrix} -\psi_1- \\ \vdots \\ -\psi_N- \end{bmatrix}$ *be the synthesis matrix* ($\Phi \in \mathbb{C}^{N \times n}$) *whose columns* $\{\varphi_1 \dots \varphi_n\}$ *form a frame for* $\mathbb{C}^N$*. Then* $\{\varphi_i\}_{i=1}^n$ *forms a tight frame with frame bound* A *if and only if* $\{\psi_i\}_{i=1}^N$ *are mutually orthogonal and* $||\psi_i||^2 = A$.

Proof. $\{\varphi_i\}_{i=1}^n$ forms a tight frame with frame bound $A \Leftrightarrow S = AI \Leftrightarrow \Phi\Phi^* = AI \Leftrightarrow$

$$
\begin{bmatrix} -\psi_1- \\ \vdots \\ -\psi_N- \end{bmatrix} \begin{bmatrix} | & & | \\ \psi_1^* & \dots & \psi_N^* \\ | & & | \end{bmatrix} = AI \Leftrightarrow \begin{bmatrix} ||\psi_1||^2 & \langle \psi_i, \psi_j \rangle & \\ & \ddots & \\ & & ||\psi_N||^2 \end{bmatrix} = AI \Leftrightarrow \langle \psi_i, \psi_j \rangle = A\delta_{ij}
$$

□

Theorem 1.3 (*Frame bound for tight frame*). *Let* $\{\varphi_i\}_{i=1}^N$ *be a tight frame for* $\mathcal{H}$ *with dim* $\mathcal{H} = N$*. Then the tight frame bound* $A = \frac{\sum_{i=1}^N ||\varphi_i||^2}{\dim \mathcal{H}}$.

Proof. Let Φ be the synthesis operator of $\{\varphi_i\}_{i=1}^N$ frame. Let $B = (\vec{e_i})_{i=1}^N$ be the standard orthonormal basis for $\mathbb{C}^N$. Then $S = \Phi\Phi^* = AI$ implies $\mathrm{tr}(\Phi\Phi^*) = \mathrm{tr}(\Phi^*\Phi) =$

$A \dim \mathcal{H}$. Now $\Phi^*\Phi : \mathbb{C}^N \to \mathbb{C}^N$, and for $(a_i) \in \mathbb{C}^N$:

$$\Phi^*\Phi(a_i) = \Phi^*\Big(\sum_{i=1}^{N} a_i \cdot \varphi_i\Big) = \sum_{i=1}^{N} a_i \,.\, \Phi^*(\varphi_i)$$

$$\Phi^*\Phi(\vec{e_i}) = \Phi^*(\varphi_i) = (\langle \varphi_i, \varphi_j \rangle)_{j=1}^{N}$$

$$[\Phi^*\Phi]_B = (\langle \varphi_i, \varphi_j \rangle)_{1 \le i,j \le N}$$

$$\sum_{i=1}^{N} ||\varphi_i||^2 = tr(\Phi^*\Phi) = A \dim \mathcal{H}.$$

□

1.2 Almost-Parseval Frame Tower

The purpose of this section is to review the construction of Fourier frames for fractal measures based on the tower construction given in Lai and Wang [3].

Definition 1.2. Let μ be a compactly supported Borel probability measure on $\mathbb{R}$. We say μ is a *frame spectral measure*, if there exists a collection of exponential functions $\{e^{2\pi i \lambda x}\}_{\lambda \in \Lambda}$ such that there exists $0 < A \le B < \infty$ with:

$$A||f||_2^2 \le \sum_{\lambda \in \Lambda} \left| \int f(x)\, e^{-2\pi i \lambda x} d\mu(x) \right|^2 \le B||f||_2^2, \ \ \forall f \in L^2(\mu). \tag{1.2}$$

Whenever such Λ exists, $\{e^{2\pi i \lambda x}\}_{\lambda \in \Lambda}$ is called a *Fourier frame* for $L^2(\mu)$ and Λ is called a *frame spectrum* (the corresponding frequency set) for μ.

Definition 1.3. Let $0 \le \epsilon_j < 1$ and $\sum_{j=1}^{\infty} \epsilon_j < +\infty$. $\{(N_j, B_j)\}_{j \in Z^+}$ is an $L^2(\mu)$ *almost-Parseval-frame tower* associated with $\{\epsilon_j\}$ if:

1. $N_j \in \mathbb{Z}^+ \setminus \{1\}$.

2. $B_j \subset \{0, 1, ..., N_j - 1\}$ and $0 \in B_j \ \forall j$.

3. Let $M_j := \#B_j$. There exists $L_j \subset \mathbb{Z}$ (with $0 \in L_j$) such that for all j:

$$(1-\epsilon_j)^2 \sum_{b\in B_j} |w_b|^2 \leq \sum_{\lambda \in L_j} \left| \frac{1}{\sqrt{M_j}} \sum_{b \in B_j} w_b e^{\frac{-2\pi i b\lambda}{N_j}} \right|^2 \leq (1+\epsilon_j)^2 \sum_{b\in B_j} |w_b|^2 \tag{1.3}$$

for all $\boldsymbol{w} = (w_b)_{b\in B_j} \in \mathbb{C}^{M_j}$.

Letting the matrix $\mathcal{F}_j = \frac{1}{\sqrt{M_j}}[e^{2\pi i b\lambda/N_j}]_{\lambda\in L_j, b\in B_j}$ and $||.||$ the standard Euclidean norm, the above equation is equivalent to:

$$(1-\epsilon_j)\ ||\boldsymbol{w}|| \leq\ ||\mathcal{F}_j \boldsymbol{w}||\ \leq (1+\epsilon_j)\ ||\boldsymbol{w}||,\ \ \forall \boldsymbol{w} \in \mathbb{C}^{M_j}.$$

Whenever $\{L_j\}_{j\in\mathbb{Z}^+}$ exists, we call $\{L_j\}_{j\in\mathbb{Z}^+}$ a *pre-spectrum* for the *almost-Parseval-frame tower*. We define the following measure to be associated with an almost-Parseval-frame tower:

$$v_j = \frac{1}{M_j} \sum_{b\in B_j} \delta_{b/N_1 N_2 \cdots N_j}$$

where δ_a denotes the Dirac measure supported on a and

$$\mu := v_1 * v_2 * v_3 * \cdots$$

$$\mu_n := v_1 * \cdots * v_n$$

$$\mu_{>n} := v_{n+1} * v_{n+2} * \cdots$$

and hence we have the decomposition $\mu = \mu_n *\ \mu_{>n}$. The support of μ is the compact set K_μ, where:

$$K_\mu := \left\{ \sum_{j=1}^{\infty} \frac{b_j}{N_1 \cdots N_j} : b_j \in B_j \right\}.$$

We denote the first n$^{\text{th}}$-partial sum of K_μ by $\boldsymbol{B_n}$ which is the support of μ_n:

$$\boldsymbol{B_n} := \frac{1}{N_1} B_1 + \frac{1}{N_1 N_2} B_2 + \cdots + \frac{1}{N_1 N_2 \cdots N_n} B_n$$

For the $\{L_j\}_{j \in \mathbf{Z}}$ in the tower, we define:

$$\boldsymbol{L_n} := L_1 + N_1 L_2 + \cdots + (N_1 N_2 \cdots N_{n-1}) L_n. \tag{1.4}$$

The fractal K_μ can be decomposed as:

$$K_\mu = \bigcup_{\mathbf{b} \in \mathbf{B_n}} (\mathbf{b} + K_{\mu,n})$$
$$K_{\mu,n} = \left\{ \sum_{j=n+1}^{\infty} \frac{b_j}{N_1 \cdots N_j} : b_j \in B_j \right\}.$$

Lemma 1.4. *Let* $f = \sum_{b \in \boldsymbol{B_n}} w_b \, \mathbb{1}_{K_b}$. *Then*

$$\int |f|^2 d\mu = \frac{1}{\boldsymbol{M_n}} \sum_{b \in B_n} |w_b|^2 \tag{1.5}$$
$$\int f(x) e^{-2\pi i \lambda x} d\mu(x) = \frac{1}{\boldsymbol{M_n}} \widehat{\mu_{>n}}(\lambda) \sum_{b \in B_n} w_b \, e^{-2\pi i \lambda b} \tag{1.6}$$

where $\boldsymbol{M_n} = M_1 \cdots M_n$

Proof. As $K_{\mathbf{b}}$ and $K_{\mathbf{b}'}$ has either empty intersection or intersects at most in one point, and $\mu(K_{\mathbf{b}}) = \dfrac{1}{\mathbf{M_n}}$, we get equation 2.5 from direct computation. Now since $\mu =$

$\mu_n * \mu_{>n}$, we have:

$$\int f(x)e^{-2\pi i\lambda x}d\mu(x) = \sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}} \int \mathbb{1}_{\mathbf{K_b}}(x)e^{-2\pi i\lambda x}d(\mu_n * \mu_{>n}(x))$$
$$= \sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}} \int \mathbb{1}_{\mathbf{b}+K_{\mu,n}}(x+y)e^{-2\pi i\lambda(x+y)}d\mu_n(x)d\mu_{>n}(y).$$

Note that $\mu_{>n}$ is supported on $K_{\mu,n}$. Thus the above is equal to:

$$= \sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}} \frac{1}{\mathbf{M_n}} \int \mathbb{1}_{\mathbf{b}+K_{\mu,n}}(\mathbf{b}+y)\, e^{-2\pi i\lambda(\mathbf{b}+y)}\, d\mu_{>n}(y)$$
$$= \sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, e^{-2\pi i\lambda\mathbf{b}} \frac{1}{\mathbf{M_n}} \int \mathbb{1}_{\mathbf{b}+K_{\mu,n}}(\mathbf{b}+y)\, e^{-2\pi i\lambda y}\, d\mu_{>n}(y)$$
$$= \frac{1}{\mathbf{M_n}} \widehat{\mu_{>n}}(\lambda) \sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, e^{-2\pi i\lambda\mathbf{b}}$$

□

Theorem 1.5. *[3] Suppose that $\{(N_j, B_j)\}$ is an almost-Parseval-frame tower associated with $\{\epsilon_j\}$ and $\{L_j\}$. Let $\boldsymbol{L_n}$ be defined as in equation 2.4, and $\Lambda := \bigcup_{n=1}^{\infty} \boldsymbol{L_n}$. If*

$$\delta(\Lambda) := \inf_n \inf_{\lambda\in \boldsymbol{L_n}} |\widehat{\mu_{>n}}(\lambda)|^2 > 0$$

then the measure μ admits a Fourier frame with frame spectrum Λ.

1.3 Existence of Almost-Parseval-Frame Tower

The following theorem shows one method for construction of an almost-Parseval-frame tower.

Theorem 1.6. *Let $N_j, M_j, K_j \in \mathbb{Z}^+$ satisfy $N_j = M_jK_j + \alpha_j$ where $0 \le \alpha_j < M_j$ and*

$$\sum_{j=1}^{\infty} \frac{\alpha_j \sqrt{M_j}}{K_j} < \infty.$$

Define:

$$B_j := \{0, K_j, \cdots, (M_j - 1)K_j\}$$

$$L_j := \{0, 1, \cdots, M_j - 1\}.$$

Then (N_j, B_j) forms an almost-Parseval-frame tower associated with:

$$\epsilon_j = \frac{2\pi\alpha_j\sqrt{M_j}}{K_j}$$

and pre-spectrum $\{L_j\}$.

Proof. Let $\mathcal{F}_j := \frac{1}{\sqrt{M_j}}\left[e^{2\pi ib\lambda/N_j}\right]_{\lambda\in L_j, b\in B_j}, \mathcal{H}_j = \frac{1}{\sqrt{M_j}}\left[e^{2\pi ib\lambda/M_jK_j}\right]_{\lambda\in L_j, b\in B_j}$.

Lemma 1.7. *$\mathcal{H}_n$ is a unitary matrix (i.e $||\mathcal{H}_n x|| = ||x||$ and $\mathcal{H}_n\mathcal{H}_n^* = \mathcal{H}_n^*\mathcal{H}_n = I$).*

Proof. Let $b = m \in L_j$ and $\lambda = nK_j \in B_j$ for $m, n = 0, 1, \cdots M_j - 1$. Therefore $e^{2\pi ib\lambda/K_jM_j} = e^{2\pi imn/M_j}$, and hence: $\mathcal{H}_j = \frac{1}{\sqrt{M_j}}\left[e^{2\pi imn/M_j}\right]_{m,n=0,1,\cdots M_j-1}$ which is the standard Fourier matrix of order M_j and hence $\mathcal{H}_j$ is unitary. □

First, we would like to show that for any $j > 0$:

$$\|\mathcal{F}_j - \mathcal{H}_j\| \leq \frac{2\pi\alpha_j\sqrt{M_j}}{K_j}.$$

For an $n \times n$ matrix A, we define the operator norm $\|A\|$ and the Frobenius norm $\|A\|_F$ as follows:

$$\begin{aligned}
\|A\| &= \max_{\|\vec{x}\|=1} \|A\vec{x}\| \\
&= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \left|a_{i,j}\, x_j\right|^2}, \text{ where } \|\vec{x}\| = 1 \\
&\leq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \left|a_{i,j}\right|^2} = \|A\|_F.
\end{aligned}$$

Applying this fact to $\mathcal{F}_j - \mathcal{H}_j$, we have:

$$\|\mathcal{F}_j - \mathcal{H}_j\|^2 \leq \|\mathcal{F}_j - \mathcal{H}_j\|_F^2 = \frac{1}{\sqrt{M_j}} \sum_{b\in B_j}\sum_{\lambda\in L_j} \left|e^{2\pi ib\lambda/N_j} - e^{2\pi ib\lambda/M_jK_j}\right|^2.$$

Recall that for any θ_1, θ_2:

$$\left|e^{i\theta_1} - e^{i\theta_2}\right| = \left|e^{i(\theta_1-\theta_2)} - 1\right| \leq \left|\theta_1 - \theta_2\right|$$

$$\left|e^{2\pi ib\lambda/N_j} - e^{2\pi ib\lambda/M_jK_j}\right|^2 \leq \left|\frac{2\pi b\lambda}{N_j} - \frac{2\pi b\lambda}{M_jK_j}\right|^2$$

$$= 4\pi^2 \frac{b^2\lambda^2\alpha_j^2}{M_j^2K_j^2N_j^2} \text{ (by } N_j = M_jK_j + \alpha_j)$$

$$\leq 4\pi^2 \frac{M_j^2\alpha_j^2}{N_j^2} \text{ (since } b \leq M_jK_j \text{ and } \lambda \leq M_j)$$

and therefore:

$$\begin{aligned}\|\mathcal{F}_j - \mathcal{H}_j\|^2 &\le \frac{4\pi^2}{M_j}\sum_{b\in B_j}\sum_{\lambda\in L_j}\frac{M_j^2\alpha_j^2}{N_j^2}\\ &= 4\pi^2\frac{M_j^3\alpha_j^2}{N_j^2}\\ &= 4\pi^2\frac{M_j\alpha_j^2}{(K_j+\alpha_j/M_j)^2}\end{aligned}$$

Now $\alpha_j \ge 0$ implies $\|\mathcal{F}_j - \mathcal{H}_j\| \le \frac{2\pi\alpha_j\sqrt{M_j}}{K_j}$. The first two conditions for the almost-Parseval-frame tower are clearly satisfied by $\{(N_j, B_j)\}$. To prove that the last condition is met, recall that $\mathcal{H}_j$ is a unitary matrix, and by the triangle inequality we have:

$$\begin{aligned}\|\mathcal{F}_j\boldsymbol{w}\| &\le \|\mathcal{H}_j\boldsymbol{w}\| + \|\mathcal{F}_j - \mathcal{H}_j\|\;\|\boldsymbol{w}\|\\ &\le \left(1+\frac{2\pi\alpha_j\sqrt{M_j}}{K_j}\right)\|\boldsymbol{w}\| = (1+\epsilon_j)\;\|\boldsymbol{w}\|.\end{aligned}$$

And finally using the reverse triangle inequality, the lower bound is established which completes the proof:

$$\begin{aligned}\|\mathcal{F}_j\boldsymbol{w}\| &\ge \|\mathcal{H}_j\boldsymbol{w}\| - \|\mathcal{F}_j - \mathcal{H}_j\|\;\|\boldsymbol{w}\|\\ &\ge \left(1-\frac{2\pi\alpha_j\sqrt{M_j}}{K_j}\right)\|\boldsymbol{w}\| = (1-\epsilon_j)\;\|\boldsymbol{w}\|.\end{aligned}$$

□

Example 1.1. Let p be an odd prime and suppose that $N_j = p^j$. Let $M_j = 2\ \forall j$.

Then $N_j = 2K_j + 1$ for some $K_j \in \mathbb{Z}^+$:

$$\sum_{j=1}^{\infty} \frac{\alpha_j \sqrt{M_j}}{K_j} = \sum_{j=1}^{\infty} \frac{1 \, . \, \sqrt{2}}{K_j} = \sum_{j=1}^{\infty} \frac{2\sqrt{2}}{p^j - 1} < \infty.$$

Thus $N_j = p^j$ and $B_j = \{0, K_j\}$ forms an almost-Parseval-frame tower with pre-spectrum $L_j = \{0, 1\}$, $\forall j$.

The following lemma translates a frame in the vector space to a Fourier frame in $L^2(\mu_n)$.

Lemma 1.8. *For any $n \geq 1$, let $\mathbf{M_n} = \prod_{j=1}^{n} M_j$. Then:*

$$\left(\prod_{j=1}^{n} (1 - \epsilon_j) \right)^2 \|\mathbf{w}\|^2 \leq \sum_{\boldsymbol{\lambda} \in \mathbf{L_n}} \left| \frac{1}{\sqrt{\mathbf{M_n}}} \sum_{\mathbf{b} \in \mathbf{B_n}} w_{\mathbf{b}} \, e^{-2\pi i \mathbf{b} \boldsymbol{\lambda}} \right|^2 \leq \left(\prod_{j=1}^{n} (1 + \epsilon_j) \right)^2 \|\mathbf{w}\|^2$$

for any $\mathbf{w} = (w_{\mathbf{b}})_{\mathbf{b} \in \mathbf{B_n}} \in \mathbb{C}^{M_1 \cdots M_n}$.

Proof. We prove the lemma by induction. For the base case $n = 1$, the statement is the almost-Parseval condition for (N_1, B_1), and is therefore trivially true. Assume now that the inequality is true for $n - 1$. Now $\mathbf{b} \in \mathbf{B_n}$ and $\boldsymbol{\lambda} \in \mathbf{L_n}$ can be decomposed as follows:

$$\boldsymbol{b} = \frac{1}{N_1 \cdots N_n} \, b_n + \boldsymbol{b_{n-1}}$$

$$\boldsymbol{\lambda} = \boldsymbol{\lambda_{n-1}} + N_1 \cdots N_{n-1} \, l_n,$$

where $b_n \in B_n, \boldsymbol{b_{n-1}} \in \boldsymbol{B_{n-1}}, \boldsymbol{\lambda_{n-1}} \in \boldsymbol{L_{n-1}}, l_n \in L_n$. Now by decomposition we have

$$\sum_{\boldsymbol{\lambda} \in \mathbf{L_n}} \left| \frac{1}{\sqrt{\mathbf{M_n}}} \sum_{\mathbf{b} \in \mathbf{B_n}} w_{\mathbf{b}} \, e^{-2\pi i \mathbf{b} \boldsymbol{\lambda}} \right|^2 =$$

$$\sum_{\lambda_{n-1}\in \mathbf{L}_{n-1}} \sum_{l_n\in L_n} \left| \sum_{b_{n-1}\in B_{n-1}} \sum_{b_n\in B_n} \frac{1}{\sqrt{\mathrm{M_n}}} w_{b_{n-1}b_n} \, e^{-2\pi i \left(\frac{1}{N_1\cdots N_n} b_n + b_{n-1}\right)\left(\lambda_{n-1} + N_1\cdots N_{n-1}\, l_n\right)} \right|^2 .$$

Note that $b_{n-1} \,.\, (N_1 \cdots N_{n-1})\; l_n \in \mathbb{Z}$, therefore the right hand side above can be written as:

$$\sum_{\lambda_{n-1}\in \mathbf{L}_{n-1}} \sum_{l_n\in L_n}$$

$$\left| \frac{1}{\sqrt{M_n}} \sum_{b_n\in B_n} \left(\sum_{b_{n-1}\in B_{n-1}} \frac{1}{\sqrt{\mathrm{M_{n-1}}}} w_{b_{n-1}b_n} \, e^{-2\pi i \left(\frac{1}{N_1\cdots N_n} b_n + b_{n-1}\right)\lambda_{n-1}} \right) e^{(-2\pi i\, b_n l_n)/N_n} \right|^2 .$$

Note that the entire content of what is inside the parentheses above can be treated as a vector, and therefore using the almost-Parseval-frame condition for (N_n, B_n) and also by the induction hypothesis, the above term is:

$$\leq (1+\epsilon_n)^2 \sum_{\lambda_{n-1}\in \mathbf{L}_{n-1}} \sum_{b_n\in B_n} \left| \left(\sum_{b_{n-1}\in B_{n-1}} \frac{1}{\sqrt{\mathrm{M_{n-1}}}} w_{b_{n-1}b_n} \, e^{-2\pi i \left(\frac{1}{N_1\cdots N_n} b_n + b_{n-1}\right)\lambda_{n-1}} \right) \right|^2$$

$$= (1+\epsilon_n)^2 \sum_{\lambda_{n-1}\in \mathbf{L}_{n-1}} \sum_{b_n\in B_n} \left| e^{-2\pi i \left(\frac{1}{N_1\cdots N_n} b_n \lambda_{n-1}\right)} \left(\sum_{b_{n-1}\in B_{n-1}} \frac{1}{\sqrt{\mathrm{M_{n-1}}}} w_{b_{n-1}b_n} \, e^{-2\pi i\, b_{n-1}\lambda_{n-1}} \right) \right|^2$$

$$= (1+\epsilon_n)^2 \sum_{b_n\in B_n} \sum_{\lambda_{n-1}\in \mathbf{L}_{n-1}} \left| \sum_{b_{n-1}\in B_{n-1}} \frac{1}{\sqrt{\mathrm{M_{n-1}}}} w_{b_{n-1}b_n} \, e^{-2\pi i\, b_{n-1}\lambda_{n-1}} \right|^2$$

$$\leq \left(\prod_{j=1}^{n}(1+\epsilon_j)\right)^2 \sum_{b_n\in B_n} \sum_{b_{n-1}\in B_{n-1}} \left|w_{b_{n-1}b_n}\right|^2$$

$$= \left(\prod_{j=1}^{n}(1+\epsilon_j)\right)^2 \|\boldsymbol{w}\|^2,$$

which completes the proof for the upper bound, and the proof for lower bound is analogous. □

Theorem 1.9. *Suppose that (N_j, B_j) is an almost Parseval frame tower and μ is the associated measure. Let L be the associated spectrum for the tower and $\delta(\Lambda) > 0$. Then μ admits a Fourier frame $E(\Lambda)$ with the lower and upper bounds of $\delta(\Lambda)\left(\prod_{j=1}^{\infty}(1-\epsilon_j)\right)^2$ and $\left(\prod_{j=1}^{\infty}(1+\epsilon_j)\right)^2$ respectively.*

Proof. To ensure that the Fourier frame inequality holds, it suffices to show that it is true for a dense set of functions in $L^2(\mu)$ (by Lemma 5.1.7 of [5]), which we will check for step functions in $\mathcal{S}$. Let $f = \sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, \mathbb{1}_{K_{\mathbf{b}}} \in \mathcal{S}_{\mathbf{n}}$. By Lemma 2.4, we have

$$\sum_{\lambda\in\mathbf{L_n}} \left|\int f(x)\, e^{-2\pi i\lambda x}\, d\mu(x)\right|^2 = \sum_{\lambda\in\mathbf{L_n}} \left|\frac{1}{\mathbf{M_n}}\widehat{\mu_{>n}}(\lambda) \sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, e^{-2\pi i\lambda\mathbf{b}}\right|^2$$

$$= \frac{1}{\mathbf{M_n}} \sum_{\lambda\in\mathbf{L_n}} |\widehat{\mu_{>n}}(\lambda)|^2 \left|\frac{1}{\sqrt{\mathbf{M_n}}} \sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, e^{-2\pi i\lambda\mathbf{b}}\right|^2.$$

Since $\delta(\Lambda) \leq |\widehat{\mu_{>n}}(\lambda)|^2 \leq 1$, using Lemma 2.8, we have

$$\frac{1}{\mathbf{M_n}}\delta(\Lambda)\left(\prod_{j=1}^{\infty}(1-\epsilon_j)\right)^2 ||\mathbf{w}||^2 \leq \sum_{\lambda\in\mathbf{L_n}} \left|\int f(x)\, e^{-2\pi i\lambda x}\, d\mu(x)\right|^2 \leq \frac{1}{\mathbf{M_n}}\left(\prod_{j=1}^{\infty}(1+\epsilon_j)\right)^2 ||\mathbf{w}||^2$$

Using Lemma 2.4 we obtain

$$\delta(\Lambda)\left(\prod_{j=1}^{\infty}(1-\epsilon_j)\right)^2\int|f|^2\,d\mu\le\sum_{\lambda\in\mathbf{L_n}}\left|\int f(x)\,e^{-2\pi i\lambda x}\,d\mu(x)\right|^2\le\left(\prod_{j=1}^{\infty}(1+\epsilon_j)\right)^2\int|f|^2\,d\mu.$$

To complete the proof, we note that for all $m > n, f \in \mathcal{S}_\mathbf{n} \subset \mathcal{S}_\mathbf{m}$, the above inequality can be written as

$$\delta(\Lambda)\left(\prod_{j=1}^{m}(1-\epsilon_j)\right)^2\int|f|^2\,d\mu\le\sum_{\lambda\in\mathbf{L_m}}\left|\int f(x)\,e^{-2\pi i\lambda x}\,d\mu(x)\right|^2\le\left(\prod_{j=1}^{m}(1+\epsilon_j)\right)^2\int|f|^2\,d\mu$$

for all $f \in \mathcal{S}_\mathbf{n}$. Taking m to infinity we get

$$\delta(\Lambda)\left(\prod_{j=1}^{\infty}(1-\epsilon_j)\right)^2\int|f|^2\,d\mu\le\sum_{\lambda\in\Lambda}\left|\int f(x)\,e^{-2\pi i\lambda x}\,d\mu(x)\right|^2\le\left(\prod_{j=1}^{\infty}(1+\epsilon_j)\right)^2\int|f|^2\,d\mu.$$

Note that the frame bounds are finite since $\sum_{j=1}^{\infty}\epsilon_j < \infty$. This shows the frame inequality for any $f \in \mathcal{S}$. Hence $E(\Lambda)$ is a Fourier frame for $L^2(\mu)$. □

Remark. The lower bound of Theorem 2.9 was subsequently improved by removing the $\delta(\Lambda)$ factor. Due to the highly technical nature of that proof, the interested readers are urged to see Theorem **11**.1 of [6].

We close this chapter by including a number of theorems and definitions that provide the foundation for proofs in the next chapter.

Definition 1.4. A sequence of vectors (x_n) in a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is called a *Riesz sequence* if there exist constants $0 < A \le B < +\infty$ such that

$$A\left(\sum_n|a_n|^2\right)\le\left\|\sum_n a_n x_n\right\|^2\le B\left(\sum_n|a_n|^2\right)$$

for all sequences $(a_n) \in \ell^2$. A Riesz sequence is called a *Riesz basis* if $\overline{\text{span}(x_n)} = \mathcal{H}$.

Theorem 1.10. *[5] Let $\{f_k\}_{k=1}^{\infty}$ be a frame for $\mathcal{H}$. Then the following are equivalent:*

- *$\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$.*
- *$\{f_k\}_{k=1}^{\infty}$ is an exact frame.*
- *$\{f_k\}_{k=1}^{\infty}$ is minimal. I.e. $f_j \notin \overline{span}\{f_k\}_{k \neq j}, \forall j \in \mathbb{N}$.*
- *$\{f_k\}_{k=1}^{\infty}$ and $\{S^{-1}f_k\}_{k=1}^{\infty}$ are biorthogonal.*
- *if $\sum_{k=1}^{\infty} c_k f_k = 0$ for some $\{c_k\}_{k=1}^{\infty} \in \ell^2(\mathbb{N})$, then $c_k = 0, \forall k \in \mathbb{N}$.*
- *$\{f_k\}_{k=1}^{\infty}$ is a basis.*

Theorem 1.11 (*Riesz Representation*). *Suppose V is a vector space and φ is a bounded linear functional on V. Then there is a unique vector $u \in V$ such that $\varphi(v) = \langle v, u \rangle, \forall v \in V$.*

Definition 1.5. A sequence of points (x_n) in a Banach space B is said to *converge weakly* to a point $x \in B$ if $f(x_n) \to f(x)$ for any bounded linear functional f defined on B, that is, for any f in the dual space B'. In the case where B is a Hilbert space, then, by Theorem 2.11 we have $f(.) = \langle ., y \rangle$. Therefore a sequence of points (x_n) in a Hilbert space $\mathcal{H}$ is said to converge weakly to a point $x \in \mathcal{H}$ if $\langle x_n, y \rangle \to \langle x, y \rangle$, $\forall y \in \mathcal{H}$. This notation is symbolized as $x_n \rightharpoonup x$.

Theorem 1.12. *[7] Let (x_n) be a sequence in normed space X. Then: (A) Strong convergence implies weak convergence with the same limit. (B) The converse of (A) is generally not true. (C) if $dim X < \infty$, then weak convergence implies strong convergence.*

Definition 1.6. A subset K of a Banach space is *weakly sequentially compact (w.s.c.)* if for any sequence $\{x_n\} \subset K$, there is a weakly convergent subsequence with limit $x \in K : x_{n_k} \rightharpoonup x$.

Theorem 1.13. *[8] (Banach-Alaoglu): Let X be a reflexive Banach space (i.e. $X = X^{**}$). Let $B = \{x \in X : ||x|| \leq 1\}$ be its closed unit ball. Then B is w.s.c.*

Chapter 2

New Results

2.1 Exactness and Overcompleteness

Theorem 2.1. *Let (N_j, B_j, L_j) be a frame tower with $\delta(\Lambda) > 0$, $\sum_{j=1}^{\infty} \epsilon_j < +\infty$, and $\#B_j = \#L_j$. Then $E(\Lambda) := \{e^{2\pi i \lambda x} : \lambda \in \Lambda\}$ is a Riesz basis for $L^2(\mu)$.*

Proof. Since $\delta(\Lambda) > 0$, then $E(\Lambda)$ is a frame by Theorem 2.5. To show the $E(\lambda)$ is a Riesz basis, it is sufficient to show that it is an exact frame (Theorem 2.10). Let λ_0 be an element in Λ and we need to show that $E(\Lambda \setminus \{\lambda_0\})$ is incomplete. $\lambda_0 \in \Lambda$ implies $\lambda_0 \in \Lambda_n, \forall n \geq n_0$ (since $0 \in L_j\ \forall j$). Now for $\mathbf{M_n} := \prod_{j=1}^{n} M_j$, there exists $\vec{w_\mathbf{b}}$ such that $||\vec{w_\mathbf{b}}||^2 = \mathbf{M_n}$ and $\sum_{\mathbf{b} \in \mathbf{B_n}} \vec{w_\mathbf{b}}\ e^{-2\pi i \mathbf{b} \lambda} = 0,\ \forall \lambda \in \Lambda_n \setminus \{\lambda_0\}$. Let $f_n := \sum_{\mathbf{b} \in \mathbf{B_n}} \vec{w_\mathbf{b}} \cdot \mathbb{1}_{\mathbf{K_b}}$, then by Lemma 2.4 we have $\int |f_n|^2 d\mu = \frac{1}{\mathbf{M_n}} \sum_{\mathbf{b} \in \mathbf{B_n}} |w_\mathbf{b}|^2 = 1$.

Let U be the closed unit ball of $L^2(\mu)$. Applying the Banach-Alaoglu Theorem (2.13), and after parsing the subsequence we have $f_n, f_{n+1}, \cdots \in U \Rightarrow \exists f \in U : f_n \rightharpoonup f$. Therefore showing that $\lim_{k \to \infty} \langle f_k, e_\lambda \rangle_{L^2(\mu)} = 0,\ \ \forall \lambda \in \Lambda \setminus \{\lambda_0\}$ and $f_k \rightharpoonup f \neq 0$ will prove the claim. By Lemma 2.8, for any $\mathbf{w} = (w_\mathbf{b})_{\mathbf{b} \in \mathbf{B_n}} \in \mathbb{C}^{M_1 \cdots M_n}$ we have

$$\prod_{j=1}^{n}(1-\epsilon_j)^2\,\|\mathbf{w}\|^2 \le \sum_{\lambda\in\Lambda_n}\left|\frac{1}{\sqrt{\mathbf{M_n}}}\sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, e^{-2\pi i\mathbf{b}\lambda}\right|^2 \le \prod_{j=1}^{n}(1+\epsilon_j)^2\,\|\mathbf{w}\|^2. \tag{2.1}$$

Now substituting our chosen $w_{\mathbf{b}}$ into (3.1), and keeping only the left inequality, we have

$$\prod_{j=1}^{n}(1-\epsilon_j)^2\,\mathbf{M_n} \le \frac{1}{\mathbf{M_n}}\left|\sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, e^{-2\pi i\mathbf{b}\lambda_0}\right|^2. \tag{2.2}$$

By (2.6) we have

$$\langle f_n, e_{\lambda_0}\rangle = \frac{1}{\mathbf{M_n}}\widehat{\mu_{>n}}(\lambda_0)\sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, e^{-2\pi i\mathbf{b}\lambda_0}$$

$$\left|\langle f_n, e_{\lambda_0}\rangle\right|^2 = \frac{1}{\mathbf{M_n^2}}\left|\widehat{\mu_{>n}}(\lambda_0)\right|^2\left|\sum_{\mathbf{b}\in\mathbf{B_n}} w_{\mathbf{b}}\, e^{-2\pi i\mathbf{b}\lambda_0}\right|^2$$

$$\ge \frac{1}{\mathbf{M_n^2}}\left|\widehat{\mu_{>n}}(\lambda_0)\right|^2\mathbf{M_n^2}\prod_{j=1}^{n}(1-\epsilon_j)^2$$

Since $\lim_{n\to\infty}\langle f_n, e_{\lambda_0}\rangle = \langle \lim_{n\to\infty} f_n, e_{\lambda_0}\rangle = \langle f, e_{\lambda_0}\rangle$ we have:

$$\left|\langle f, e_{\lambda_0}\rangle\right|^2 \ge \left|\widehat{\mu_{>n}}(\lambda_0)\right|^2\prod_{j=1}^{\infty}(1-\epsilon_j)^2 \ge (\delta(\Lambda))^2\,A_\infty > 0$$

where: $A_\infty := \prod_{j=1}^{\infty}(1-\epsilon_j)^2 > 0$ since $\sum_{j=1}^{\infty}\epsilon_j < +\infty$.

Hence $f \neq 0$.

□

2.2 Frame with ∞-Redundancy

Definition 2.1. A frame is said to have k-redundancy (where $k \in \mathbb{Z}^+ \cup \{\infty\}$) if there exist k elements of the frame such that if we remove all k elements the remaining elements still form a frame.

Next one possible pathway for construction of frame with ∞-redundancy is provided.

Theorem 2.2. *Assume we have an almost-Parseval-frame tower with* $\#L_1 > \#B_1$ *and* $\delta(\Lambda) > 0$. *Then the resulting Fourier frame has* ∞*-redundancy.*

Proof. $\#L_1 > \#B_1$ implies that the first level of the tower $\{(e_\lambda(b))_{b\in B_1} : \lambda \in L_1\}$ is overcomplete in the finite-dimensional space $\mathbb{C}^{\#B_1}$. Now assume you remove one element of L_1, say element l_1:

$$\widetilde{L_1} := L_1 \setminus \{l_1\}$$
$$\widetilde{\Lambda} := \widetilde{L_1} + \left\{ \sum_{j\geq 2} (N_1 \cdots N_{j-1}) l_j : l_j \in L_j \right\}.$$

By definition 3.1, $E(\Lambda)$ has ∞-redundancy since after removing infinite number of elements namely $\{l_1\} + \left\{ \sum_{j\geq 2} (N_1 \cdots N_{j-1}) l_j : l_j \in L_j \right\}$, the remaining elements meet the sufficient condition to form a frame by Theorem 2.5:

$$\delta(\widetilde{\Lambda}) \geq \delta(\Lambda) := \inf_n \inf_{\lambda \in L_n} |\widehat{\mu_{>n}}(\lambda)|^2 > 0.$$

□

Theorem 3.2 paved a potential path to create a frame with ∞-redundancy, next an explicit example is provided.

Lemma 2.3. *Let (N_j, B_j, L_j) be a frame tower where $\#B_j < \#L_j$ at every level, with:*

$$B_j = \{0, K_j, \ldots, (M_j - 2)K_j\},\ K_j \geq 2, M_j \geq 3$$
$$L_j = \{0, 1, \ldots, (M_j - 1)\}$$
$$H_j = \frac{1}{\sqrt{M_j - 1}}(e^{\frac{2\pi i \lambda b}{M_j K_j}}).$$

then the associated measure admits a tight Fourier frame.

Proof. H_j has dimension $|L_j| \times |B_j| = M_j \times (M_j - 1)$ and is therefore not a square matrix. By Theorem 2.2, rows of H_j form a tight frame since the columns are mutually orthogonal. This is because we started with a matrix H_j' whose columns were mutually orthogonal and then we created a new matrix H_j by deleting a column of H_j that corresponded to the $(M_j - 1)K_j$ term of B_j. By Theorem 2.3, H_j has a tight frame bound $A = \dfrac{\sum_{l \in Lj} \|e_l\|^2}{\dim \mathcal{H}} = \dfrac{M_j}{M_j - 1}$, and hence $\|H_j \boldsymbol{x}\|^2 = \dfrac{M_j}{M_j - 1}\|\boldsymbol{x}\|^2$, which proves that the frame is tight.

Next to complete the proof, we have to show that $\delta(\Lambda) = \inf_k \inf_{\lambda \in L_k} |\widehat{\mu_{>k}}(\lambda)|^2 > 0$. Let $\mathcal{F}$ denote the Fourier transform operator. Recall the following properties of the Fourier transform for functions f, g and the Dirac delta function δ:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \,.\, \mathcal{F}\{g\}$$
$$\mathcal{F}_x\{\delta(x - x_0)\}(k) = \int_{-\infty}^{+\infty} \delta(x - x_0) e^{-2\pi i k x} dx = e^{-2\pi i k x_0}$$

Using definitions of μ and v_j from section 2.2, and the above properties of the Fourier operator, the Fourier transform of μ is given by:

$$\widehat{\mu}(\xi) = \prod_{j=1}^{\infty} \widehat{v_j}(\xi) = \prod_{j=1}^{\infty} \left[\frac{1}{M_j - 1} \sum_{k=0}^{M_j-2} e^{-2\pi i k K_j \xi / N_1 \cdots N_j} \right]$$

$$\widehat{v_j}(\xi) = \frac{1}{M_j - 1} \sum_{k=0}^{M_j-2} (e^{-2\pi i c_j \xi})^k = \frac{1}{M_j - 1} \left[\frac{1 - (e^{-2\pi i c_j \xi})^{M_j - 1}}{1 - (e^{-2\pi i c_j \xi})} \right]$$

where $c_j := K_j / N_1 \cdots N_j$

$$\widehat{v_j}(\xi) = \frac{1}{M_j - 1} \left[\frac{(e^{\pi i c_j \xi})}{(e^{\pi i c_j \xi})} \right] \left[\frac{1 - (e^{-2\pi i c_j \xi})^{M_j - 1}}{1 - (e^{-2\pi i c_j \xi})} \right]$$

$$\widehat{v_j}(\xi) = \frac{1}{M_j - 1} \left[\frac{(e^{\pi i c_j \xi})^{2 - M_j} \ (e^{\pi i c_j \xi})^{M_j - 1}}{(e^{\pi i c_j \xi})} \right] \left[\frac{1 - (e^{-2\pi i c_j \xi})^{M_j - 1}}{1 - (e^{-2\pi i c_j \xi})} \right]$$

$$\widehat{v_j}(\xi) = \frac{1}{M_j - 1} (e^{\pi i c_j \xi})^{2 - M_j} \left[\frac{(e^{\pi i c_j \xi})^{M_j - 1} - (e^{-\pi i c_j \xi})^{M_j - 1}}{(e^{\pi i c_j \xi}) - (e^{-\pi i c_j \xi})} \right]$$

Therefore:

$$\widehat{v_j}(\xi) = \frac{1}{M_j - 1} (e^{\pi i c_j \xi})^{2 - M_j} \left[\frac{\sin \pi c_j \xi (M_j - 1)}{\sin \pi c_j \xi} \right], \text{ if } \xi \notin \frac{1}{c_j}\mathbb{Z}$$

$$\widehat{v_j}(\xi) = 1, \qquad \text{if } \xi \in \frac{1}{c_j}\mathbb{Z}$$

$$|\widehat{\mu_{>k}}(\lambda)|^2 = \prod_{j=1}^{\infty} |\widehat{v_{>k+j}}(\lambda)|^2 = \prod_{j=1}^{\infty} \left| \frac{1}{M_{k+j} - 1} \sum_{k=0}^{M_{k+j}-2} e^{-2\pi i k K_{k+j} \lambda / (N_1 \cdots N_k N_{k+1} \cdots N_{k+j})} \right|^2 \tag{2.3}$$

For any $\lambda \in \boldsymbol{L_k}$ for which the terms $|\widehat{v_{>k+j}}(\lambda)|^2 < 1$, we have:

$$\left|\frac{1}{M_{k+j}-1}\sum_{k=0}^{M_{k+j}-2} e^{-2\pi i k K_{k+j}\lambda/(N_1\cdots N_k N_{k+1}\cdots N_{k+j})}\right|^2 = \left|\frac{1}{M_{k+j}-1}\frac{\sin \pi c_{k+j}(M_{k+j}-1)\lambda}{\sin \pi c_{k+j}\lambda}\right|^2$$

Using estimates of $\sin x \leq x$ and $\sin x \geq x - x^3/3!$, we have:

$$\left|\frac{1}{M_{k+j}-1}\frac{\sin \pi c_{k+j}(M_{k+j}-1)\lambda}{\sin \pi c_{k+j}\lambda}\right|^2 \geq \left|\frac{\sin \pi c_{k+j}(M_{k+j}-1)\lambda}{\pi c_{k+j}(M_{k+j}-1)\lambda}\right|^2 \geq \left(1-\frac{(\pi c_{k+j}(M_{k+j}-1)\lambda)^2}{3!}\right)^2$$

Recall that $c_{k+j} = K_{k+j}/N_1\cdots N_{k+j}$, therefore:

$$\left(1-\frac{(\pi c_{k+j}(M_{k+j}-1)\lambda)^2}{3!}\right)^2 > \left(1-\frac{(\pi c_{k+j}M_{k+j}\lambda)^2}{3!}\right)^2 \tag{2.4}$$

$$= \left(1-\frac{\pi^2\lambda^2 M_{k+j}^2 K_{k+j}^2}{6N_1^2\cdots N_{k+j}^2}\right)^2 \tag{2.5}$$

$$= \left(1-\left(\frac{\pi^2}{6N_{k+1}^2\cdots N_{k+j-1}^2}\right)\left(\frac{M_{k+j}K_{k+j}}{N_{k+j}}\right)^2\left(\frac{\lambda}{N_1\cdots N_k}\right)^2\right)^2 \tag{2.6}$$

Now $N_i = M_iK_i + \alpha_i$ and $L_j = \{0, 1, \ldots, (M_j - 1)\}$, thus $l_i \leq M_i - 1 < N_i$

$M_i \geq 3, K_i \geq 2 \Rightarrow N_i \geq 6$

For $\lambda \in \boldsymbol{L_k} : \lambda = l_1 + N_1 l_2 + \cdots + (N_1 \cdots N_{k-1})l_k$ for some $l_i \in L_i$, and thus:

$$\begin{aligned}
\frac{\lambda}{N_1\cdots N_k} &= \frac{l_1 + N_1 l_2 + \cdots + (N_1\cdots N_{k-1}) l_k}{N_1\cdots N_k} \\
&= \frac{l_k}{N_k} + \frac{l_{k-1}}{N_k N_{k-1}} + \cdots + \frac{l_2}{N_k\cdots N_2} + \frac{l_1}{N_k\cdots N_1} \\
&< \frac{M_k}{N_k} + \frac{M_{k-1}}{N_k N_{k-1}} + \cdots + \frac{M_1}{N_k\cdots N_1} \\
&\le \frac{1}{K_k} + \frac{1}{N_k K_{k-1}} + \cdots + \frac{1}{N_k\cdots N_2 K_1} \\
&\le \frac{1}{2}\left(1 + \frac{1}{N_k} + \cdots + \frac{1}{N_k\cdots N_2}\right) \\
&\le \frac{1}{2}\sum_{j=0}^{j=k-1}(\frac{1}{6})^j \le \frac{1}{2}\sum_{j=0}^{\infty}(\frac{1}{6})^j = \frac{1}{2}\times\frac{6}{5} = \frac{3}{5}
\end{aligned}$$

Note that:

$$N_{k+j} = M_{k+j}K_{k+j} + \alpha_{k+j} \Rightarrow \frac{M_{k+j}K_{k+j}}{N_{k+j}} \le 1$$

Now for $j = 1$:

$$\text{Equation (3.8)} \ge \left(1 - (\frac{\pi^2}{6})(\frac{3}{5})^2\right)^2 = \left(1 - \frac{3\pi^2}{50}\right)^2 > 0$$

For $j > 1$:

$$\text{Equation (3.8)} \ge \left(1 - \frac{\pi^2}{6\,.\,6^{2(j-1)}}(\frac{3}{5})^2\right)^2 > 0$$

Putting it all back to (3.5), we obtain:

$$|\widehat{\mu_{>k}}(\lambda)|^2 \ge \prod_{j=1}^{\infty}\left(1 - \frac{3\pi^2}{50\,.\,6^{2(j-1)}}\right)^2 := c_0$$

Since $\sum_{j=1}^{\infty} \frac{3\pi^2}{50 \cdot 6^{2(j-1)}} < +\infty$ implies $c_0 > 0$, by Theorem 2.5, μ admits a Fourier frame with spectrum Λ. $\square$

Chapter 3

Appendix

Here a number fundamental definitions and concepts that are used throughout this thesis are included:

Definition 3.1. A metric space is a pair (X, d), where X is a set and d is a metric (or distance function) on X that is a function on $X \times X$ such that $\forall x, y, z \in X$, we have:

- (M1) d is real-valued, finite and nonnegative
- (M2) $d(x, y) = 0 \Longleftrightarrow x = y$
- (M3) $d(x, y) = d(y, x)$
- (M4) $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 3.2. A sequence (x_n) in a metric space $X = (X, d)$ is said to converge if

$$\exists x \in X : \lim_{n \to \infty} d(x_n, x) = 0$$

$x_n \longrightarrow x$ notation is also used commonly to denote convergence of a sequence.

Definition 3.3. A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy if $\forall \varepsilon > 0$, $\exists N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n > N$. The space X is said to be complete if every Cauchy sequence in X converges to an element in X.

Definition 3.4. A norm on a vector space X is a real-valued function on X, whose value at an $x \in X$ is denoted by $||x||$ with the following properties for $x, y, \in X, \alpha \in \mathbb{F}$:

- (N1) $||x|| \geq 0$
- (N2) $||x|| = 0 \Longleftrightarrow x = 0$
- (N3) $||\alpha x|| = |\alpha| \; ||x||$
- (N4) $||x + y|| \leq ||x|| + ||y||$

A norm on X defines a metric d on X which is given by $d(x, y) = ||x - y||$ and is called the metric induced by the norm.

Definition 3.5. A Hilbert space is a Banach space with additional geometric properties. In particular, the norm of the Hilbert space is obtained from an inner product that mimics the properties of dot product of vectors in $\mathbb{C}^n$. A vector space $\mathcal{H}$ is an inner product space if for each $x, y \in \mathcal{H}$ there exists a scalar $\langle x, y \rangle$ such that:

- (IP1) $\langle x, y \rangle$ is real and $\langle x, y \rangle \geq 0$, $\forall x \in \mathcal{H}$
- (IP2) $\langle x, x \rangle \; = 0 \Longleftrightarrow x = 0$
- (IP3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- (IP4) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

With norm defined as $||.||$ by $||x|| = \sqrt{\langle x, x \rangle}$, with orthogonality defined by $\langle x, y \rangle = 0$. Hilbert space theory is much richer than normed and Banach spaces, with the following distinguishing features: (A) Representation of $\mathcal{H}$ as a direct sum of closed subgroups and its orthogonal complements (B) Orthonormal sets and sequences the corresponding representation of elements of $\mathcal{H}$ (C) Hilbert-adjoint operator T^* of bounded linear operator T (D) Riesz representation of bounded linear functional by inner products.

Lemma 3.1. *[9] If $U : X \longrightarrow X$ is bounded and $||I - U|| < 1$, then U is invertible and $U^{-1} = \sum_{k=0}^{\infty} (I - U)^k$. Furthermore $||U^{-1}|| \leq \dfrac{1}{1 - ||I - U||}$.*

Bibliography

[1] Duffin and A. Schaeffer. A class of nonharmonic fourier series. *Tran. Amer. Math. Soc.*, 72:341–366, 1952.

[2] P. Jorgensen and S. Pedersen. Dense analytic subspaces in fractal l^2-spaces. *J. Anal. Math.*, 75:185–228, 1998.

[3] Chun-Kit Lai and Yang Wang. Non-spectral fractal measures with fourier frames. *J. Fractal Geom.*, 4:305–327, 2017.

[4] Deguang Han, Keri Kornelson, David Larson, and Eric Weber. *Frames for Undergraduates.* AMS, 2007.

[5] Ole Christensen. *An Introduction to Frames and Riesz Bases.* Birkhauser, 2003.

[6] Dorin Dutkay; John Haussermann; Chun-Kit Lai. Hadamard triples generate self-affine spectral measures. *eprint arXiv:1607.08024*, 2016.

[7] Erwin Kreyszig. *Introductory Functional Analysis with Applications.* John Wiley & Sons Inc., 1978.

[8] Laszlo Erdos. Banach–alaoglu theorems, 2007.

[9] Christopher Heil. *A Basis Theory Primer.* Birkhauser, 2011.