ON δ-POLYNOMIALS FOR LATTICE PARALLELEPIPEDS

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by

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CERTIFICATION OF APPROVAL

I certify that I have read *ON δ-POLYNOMIALS FOR LATTICE PARALLELEPIPEDS* by Emily McCullough and that in my opinion this work meets the criteria for approving a thesis submitted in partial fulfillment of the requirements for the degree: Master of Arts in Mathematics at San Francisco State University.

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We introduce the (B, ℓ)-Eulerian numbers, a refined descent statistic on the set of signed permutations on $\{1, 2, \ldots, d\}$, and show that these numbers appear as the coefficients of the δ-polynomial (also known as the Ehrhart h-polynomial or the h^*-polynomial) for the d-dimensional half-open ± 1-cube with ℓ non-translate facets removed. Using an interplay of the geometry and combinatorics, we characterize the δ-polynomials for specific families of half-open parallelepipeds and improve upon known inequality relations on their coefficients. In particular, we prove that the coefficients of the δ-polynomial for half-open parallelepipeds with lattice centrally symmetric edges are alternatingly increasing. Our results extend naturally to the δ-polynomials for closed lattice zonotopes.

I certify that the Abstract is a correct representation of the content of this thesis.

Matthias Beck 5/18/16

Chair, Thesis Committee Date

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Chapter 1

Introduction

This thesis deals with results in the fields of algebraic and geometric combinatorics, specifically in the area of Ehrhart theory. Our main objects of study are *half-open lattice d-parallelepipeds* — higher-dimensional analogues to 2-dimensional parallelograms with integer vertices and $j \leq d$ non-translate facets removed — and the associated *δ-polynomials* (also known as h^*-polynomials or Ehrhart h-polynomials).

Using the interplay of the geometry provided by disjoint decompositions of the parallelepipeds and the combinatorics provided by relevant permutation descent statistics, we improve upon known inequality relations on the coefficients of the δ-polynomials for specific families of half-open parallelepipeds. In particular, we improve these inequalities for certain families of half-open parallelepipeds with an interior lattice point. Our results for half-open lattice parallelepipeds extend naturally to the δ-polynomials for closed lattice zonotopes.

Chapter 2 is a review of the basics. The first section is an introduction to Ehrhart theory. We start from the beginning with formal definitions for *lattice polytope* and related terminology such as *dimension, hyperplane, face, boundary, relative interior*, and various notions of *volume*. We introduce the *lattice point enumerator*, or *Ehrhart polynomial*, of a lattice polytope $\mathcal{P}$, and the related *Ehrhart series* of $\mathcal{P}$. The numerator of the Ehrhart series in its rational form is the *δ-polynomial* of $\mathcal{P}$. In this section we also define what it means for a δ-polynomial to be *unimodal* or *alternatingly increasing*. These inequality relations on the coefficients of the δ-polynomial are the focus of this thesis.

In the second section of Chapter 2 we introduce the main geometric objects of our study: *lattice parallelepipeds*, in particular *unit cubes*, and *lattice zonotopes* — all *integrally closed* lattice polytopes. We consider a theorem due to Shephard which depicts the close relationship between zonotopes and parallelepipeds [11]. We also note the close relationship between subdivisions and dissections of a polytope. In this way we set up and motivate our later results on the δ-polynomials for zonotopes.

This thesis is part of a joint project with Matthias Beck and Katharina Jochemko. Chapter 3, with the exception of Lemma 3.4, is a reformulation of Jochemko's work on zonotopes in [8]. It is the starting point for our extensions and contributions, which appear in Chapters 4 and 5.

A conjecture by Stanley [13] infers that the coefficients of the δ-polynomial for every integrally closed lattice polytope are unimodal. Schepers and Van Langenhoven [10] prove that this weakening of Stanley's conjecture holds for (closed) lattice parallelepipeds[1], the simplest example of an integrally closed lattice polytope after *unimodular simplices*. Jochemko [8] extends their work to half-open lattice parallelepipeds by first interpreting the A-polynomials of [10] in terms of refined descent statistics on S_d — statistics we call the (A, j)*-Eulerian numbers*. Using the symmetric and recursive properties of the (A, j)-Eulerian numbers provided by Brenti and Welker [5, Lemma 2.5], Jochemko proves the (A, j)-Eulerian numbers are unimodal and specifies the peak according to d and j. Jochemko further provides a characterization of the δ-polynomial for half-open parallelepipeds as a nonnegative linear combination of (A, j)-Eulerian polynomials. Together these results imply that the coefficients of the δ-polynomials for half-open parallelepipeds are unimodal. Using the aforementioned theorem of Shephard, combined with the Beneath and Beyond construction of Köppe and Verdoolaege [9], Jochemko further extends the unimodality results to the δ-polynomials for lattice zonotopes — a (large) family of integrally closed lattice polytopes. In Chapter 3, we carefully build up to these results. Additionally, through an extension of Jochemko's unimodality proof, we show that the (A, j)-Eulerian numbers are alternatingly increasing for sufficiently large j.

Chapter 4 deals with the relationship between the (B, ℓ)*-Eulerian numbers*, a refined descent statistic on B_d, the set of *signed permutations* on $[d]$, and the δ-polynomial for $[-1, 1]^d_\ell$, the d-dimensional *half-open* ± 1*-cube* with ℓ non-translate facets removed. Our work is motivated by a result due to Brenti [4, Theorem 3.4] as seen from the geometric perspective presented by Beck and Braun in [1]. The result

[1]Stanley's original conjecture has been disproved in its general form.

(for $q = 1$) says that the type-B Eulerian polynomial $B(d, t)$ is the δ-polynomial for the (closed) ± 1-*cube* $[-1, 1]^d$. In Section 4.1 we characterize $B(d, t)$ as a positive linear combination of (A, j)-Eulerian polynomials of the same degree. We do this via a disjoint decomposition of $[-1, 1]^d$ into *unit cells* U_I^d where $I \subseteq [d]$, geometric objects which are congruent to unit d-cubes. Using the symmetric properties of the (A, j)-Eulerian numbers [5, Lemma 2.5], we give a geometric proof that the coefficients of $B(d, t)$ are symmetric and unimodal, and thus alternatingly increasing. This is not a new result. It follows from [4, Theorem 2.4] and can also be deduced from [10, Proposition 2.17].

In Section 4.2 we extend the methods of [1] to half-open ± 1-cubes. We consider a disjoint decomposition of $[-1, 1]_\ell^d$ into half-open unimodular simplices indexed by signed permutations $(\pi, \epsilon) \in B_d$ and show that the removed facets of the simplex $\triangle_{(\pi, \epsilon)}^{d, \ell}$ are enumerated by our refined descent statistic. In this way we prove that the (B, ℓ)-Eulerian polynomial $B_{\ell+1}(d + 1, t)$ is the δ-polynomial for $[-1, 1]_\ell^d$.

We further prove that the (B, ℓ)-Eulerian polynomial is alternatingly increasing. As before this is seen via a disjoint decomposition of $[-1, 1]_\ell^d$ into unit cells. From the decomposition we obtain a characterization of the δ-polynomial for $[-1, 1]_\ell^d$ as a linear combination of (A, j)-Eulerian polynomials of the same degree. The non-negative integral coefficients of this linear combination have symmetric, unimodal and recursive properties which we use to show that $B_{\ell+1}(d + 1, t)$ is alternatingly increasing. Our geometric perspective results in an alternative geometric proof for the palindromic and unimodal properties of the binomial coefficients as well as a geometric interpretation of the well-known recursive formula

$$\binom{d}{j} = \binom{d-1}{j} + \binom{d-1}{j-1}.$$

In Chapter 5 we consider the δ-polynomials for lattice parallelepipeds and zonotopes with lattice centrally symmetric edges. An edge of a polytope is *lattice centrally symmetric* if the midpoint of that edge is a lattice point. Using the symmetry of the edges as well as results of Jochemko and Schepers and Van Langenhoven seen in Chapter 3, we express the δ-polynomial for half-open parallelepipeds with lattice centrally symmetric edges in terms of the δ-polynomials for half-open ± 1-cubes of the same dimension. In particular, we provide a characterization for the δ-polynomial as a nonnegative linear combination of (B,ℓ)-Eulerian polynomials of the same degree. Applying our results about the inequality relations on these polynomials from Chapter 4, we see that the δ-polynomial for half-open parallelepipeds with lattice centrally symmetric edges is alternatingly increasing. We further extend the alternatingly increasing results to the δ-polynomials for closed lattice zonotopes with lattice centrally symmetric edges using [11, Theorems 54 and 56] and [8, Corollary 3.5.3].

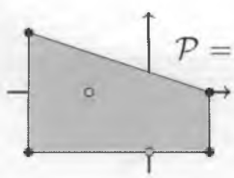
We conclude with a short discussion of extensions and open questions.

Chapter 2

Basics

2.1 An Introduction to Lattice Polytopes and Ehrhart Theory

Consider a finite set of points in $\mathbb{R}^m$: $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Imagine shrink-wrapping these points in m-space. The object you obtain, called a **polytope**, is a d-dimensional analogue of a 2-dimensional convex polygon. Like a polygon, a polytope is a closed and convex geometric object with flat faces and extreme points called **vertices**. The vertices are necessarily some subset of the original set of points. See Figure 2.1.



$$\mathcal{P} = \operatorname{conv}\{(-2,1), (-2,-1), (-1,0), (0,-1), (1,0), (1,-1)\}$$

Figure 2.1: A lattice polytope $\mathcal{P}$. The vertices appear as solid points.

Formally we define a polytope $\mathcal{P}$ as the convex hull of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$:

$$\mathcal{P} = \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} := \left\{ \sum_{i\in[n]} \lambda_i \mathbf{v}_i : \lambda_i \geq 0 \text{ and } \sum_{i\in[n]} \lambda_i = 1 \right\}.$$

A **lattice polytope**[1] is the convex hull in $\mathbb{R}^m$ of finitely many **lattice points** — points in the **integer lattice** $\mathbb{Z}^m$. We are interested in lattice polytopes exclusively, so from here on we shall take the term *polytope* to mean *lattice polytope* while omitting the descriptor. We are also interested in positive integer dilates of $\mathcal{P}$, denoted

$$n\mathcal{P} := \{n\mathbf{x} : \mathbf{x} \in \mathcal{P}\}.$$

The **(affine) span** of polytope $\mathcal{P}$,

$$\operatorname{span}(\mathcal{P}) := \{\mu\mathbf{x} + \lambda\mathbf{y} : \mathbf{x}, \mathbf{y} \in \mathcal{P} \text{ and } \mu + \lambda = 1\},$$

is the translate of a vector space, called an **affine space**. The dimension of an affine space is equal to the dimension of the vector space of which it is a translate. We define the **dimension** of $\mathcal{P}$ as the dimension of $\operatorname{span}(\mathcal{P})$ [2]. If $\mathcal{P}$ has dimension d, we call $\mathcal{P}$ a d**-polytope** and write $\dim(\mathcal{P}) = d$.

A **hyperplane** in $\mathbb{R}^m$ is a set of the form

$$H = \{\mathbf{x} \in \mathbb{R}^m : \boldsymbol{a} \cdot \mathbf{x} = b\} \text{ for some } \boldsymbol{a} \in \mathbb{R}^m \text{ and } b \in \mathbb{R}.$$

[1] A lattice polytope is also known as an integral polytope.

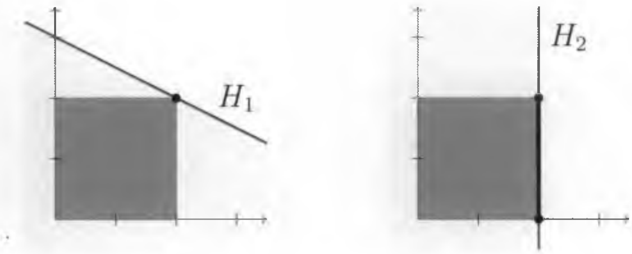


Figure 2.2: Supporting hyperplanes H_1 and H_2 in $\mathbb{R}^2$ define a vertex and a facet (also an edge) of polytope $\mathcal{P} = \operatorname{conv}\{(0,0),(2,0),(2,2),(0,2)\}$.

When $\boldsymbol{a}$ is the zero vector and $b = 0$, $H = \mathbb{R}^m$. When this is not the case, the dimension of H is one less than the dimension of the ambient space. Just as a line (a hyperplane in $\mathbb{R}^2$) divides $\mathbb{R}^2$ into two spaces, the hyperplane H divides $\mathbb{R}^m$ into two spaces, called **halfspaces**. We denote the halfspaces by

$$H^{\leq} := \{\mathbf{x} \in \mathbb{R}^m : \boldsymbol{a} \cdot \mathbf{x} \leq b\} \text{ and}$$
$$H^{\geq} := \{\mathbf{x} \in \mathbb{R}^m : \boldsymbol{a} \cdot \mathbf{x} \geq b\}.$$

We call H a **supporting hyperplane** of the polytope $\mathcal{P}$ if $\mathcal{P}$ lies entirely in one halfspace of H [2]. See Figure 2.2.

A **face** $\mathcal{F}$ of a polytope $\mathcal{P}$ is a subset of the polytope defined by the intersection of $\mathcal{P}$ with one of its supporting hyperplanes. That is, $\mathcal{F} = \mathcal{P} \cap H$, where H is a supporting hyperplane of $\mathcal{P}$. When $H = \mathbb{R}^m$ we have

$$\mathcal{F} = \mathcal{P} \cap H = \mathcal{P} \cap \mathbb{R}^m = \mathcal{P},$$

Figure 2.3: The relative volume of the blue facet of the 3-polytope is equal to 1.

meaning $\mathcal{P}$ is a face of $\mathcal{P}$. When H is a supporting hyperplane that does not meet $\mathcal{P}$, then $\mathcal{P} \cap H$ is empty, meaning $\mathcal{F} = \emptyset$ is a face of $\mathcal{P}$. The faces of a polytope are themselves polytopes [7]. $\mathcal{P}$ and $\emptyset$ are the trivial faces of $\mathcal{P}$ of dimension d and -1, respectively. The faces of $\mathcal{P}$ of dimension 0 are called **vertices**, the faces of $\mathcal{P}$ of dimension 1 are called **edges**, and the faces of $\mathcal{P}$ of dimension $d-1$ are called **facets**. A face of $\mathcal{P}$ of dimension r is called an r-**face**. When $r < d$, we say r-face $\mathcal{F}$ is a **proper** face of $\mathcal{P}$.

We distinguish between the **boundary** and the **relative interior** of a polytope $\mathcal{P}$ in the following manner. The boundary of $\mathcal{P}$ consists of those points contained in some proper face of $\mathcal{P}$ and the relative interior of $\mathcal{P}$ (denoted by $\mathcal{P}^\circ$) consists of those points in $\mathcal{P}$ that are *not* contained in the boundary.

The **relative volume** of polytope $\mathcal{P}$ is the Euclidean volume of $\mathcal{P}$ relative to its affine span [2]. This definition of volume allows us to assign a non-zero measure of volume to objects that are not full-dimensional, such as the proper faces of a polytope. Consider the blue facet of the 3-polytope in Figure 2.3 for example. The span of the blue facet is represented in yellow; it is a 2-dimensional hyperplane. We see that the blue facet is a unit square relative to its span. Therefore, the blue facet has relative volume 1.

We further define the **normalized volume** of a d-polytope $\mathcal{P}$ to be $d!$ times the relative volume of $\mathcal{P}$ and write $\mathrm{vol}(\mathcal{P})$ to denote this value [10]. Normalized volume changes the unit of volume measure from the d-dimensional unit cube to a d-dimensional unimodular simplex. We describe these objects in detail below.

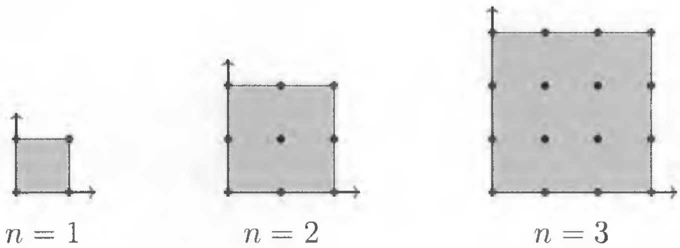
A fundamental combinatorial object in Ehrhart theory is the **lattice point enumerator** of a polytope. The lattice point enumerator of $\mathcal{P}$ counts the number of lattice points in the n^{th} positive integer dilate of $\mathcal{P}$ and is denoted by

$$\mathsf{ehr}(\mathcal{P}, n) := \#\left(n\mathcal{P} \cap \mathbb{Z}^m\right) .$$

This counting function for lattice polytopes is a polynomial in n of degree d [6]. We observe this in a simple example in Figure 2.4. The result is due to Eugène Ehrhart and accordingly, $\mathsf{ehr}(\mathcal{P}, n)$ is called the **Ehrhart polynomial** of $\mathcal{P}$. When embedded in a generating function, we obtain the **Ehrhart series** of $\mathcal{P}$:

$$\mathsf{Ehr}(\mathcal{P}, t) := 1 + \sum_{n \geq 1} \mathsf{ehr}(\mathcal{P}, n) t^n .$$

The Ehrhart series of every lattice polytope can be written as a rational function in



$$\mathsf{ehr}(\mathcal{P}, 1) = 4$$
$$\mathsf{ehr}(\mathcal{P}, 2) = 9$$
$$\mathsf{ehr}(\mathcal{P}, 3) = 16$$
$$\mathsf{ehr}(\mathcal{P}, n) = (n+1)^2$$

Figure 2.4: Observing the Ehrhart polynomial of the 2-cube.

the following form:

$$\frac{\delta(\mathcal{P}, t)}{(1-t)^{d+1}} = \frac{\delta_0 + \delta_1 t + \cdots + \delta_d t^d}{(1-t)^{d+1}}, \tag{2.1}$$

where the δ_i are nonnegative integers and $\delta_0 = 1$ [2, 12]. The numerator $\delta(\mathcal{P}, t)$ is called the **δ-polynomial** of $\mathcal{P}$ and the sequence of coefficients $(\delta_0, \delta_1, \ldots, \delta_d)$ is called the **δ-vector**[2] of $\mathcal{P}$. A property of this polynomial (vector) with particular significance to Ehrhart theory is the fact that the normalized volume of a polytope $\mathcal{P}$ is equal to the sum of the δ_i [2]; that is,

$$\mathrm{vol}(\mathcal{P}) = \delta_0 + \delta_1 + \cdots + \delta_d \,. \tag{2.2}$$

Specific relationships among these coefficients for particular families of polytopes are the focus of this thesis. One property we are interested in is unimodality. A δ-vector is **unimodal** if it has a single **peak**; that is, if its entries increase up to some point, then decrease. Symbolically, $(\delta_0, \delta_1, \ldots, \delta_d)$ is unimodal if there exists a

[2]Other names for the δ-polynomial (vector) are Ehrhart h-polynomial (vector) and h^*-polynomial (vector).

$k \in \{0, 1, \ldots, d\}$ such that

$$\delta_0 \leq \delta_1 \leq \cdots \leq \delta_k \geq \cdots \geq \delta_d\,.$$

We are also interested in whether a δ-vector is **alternatingly increasing**; that is, if

$$\delta_0 \leq \delta_d \leq \delta_1 \leq \cdots \leq \delta_{\lfloor \frac{d+1}{2} \rfloor}\,.$$

This is a stronger inequality property than unimodality: If $(\delta_0, \delta_1, \ldots, \delta_d)$ is alternatingly increasing, then $(\delta_0, \delta_1, \ldots, \delta_d)$ is unimodal with peak at $\lfloor \frac{d+1}{2} \rfloor$. We say a δ-polynomial is unimodal or alternatingly increasing if the corresponding δ-vector is.

2.2 Parallelepipeds, Unit Cubes and Zonotopes

Given d linearly independent generating vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{Z}^m$, we define the d-dimensional **(lattice) parallelepiped**

$$\Diamond(\mathbf{v}_1, \ldots, \mathbf{v}_d) := \left\{ \sum_{i \in [d]} \lambda_i \mathbf{v}_i : 0 \leq \lambda_i \leq 1 \right\}.$$

A zero-dimensional parallelepiped is a point. A one-dimensional parallelepiped is a line segment. A two-dimensional parallelepiped is a parallelogram. A three-dimensional parallelepiped (known by this name in layperson's terms as well) is a

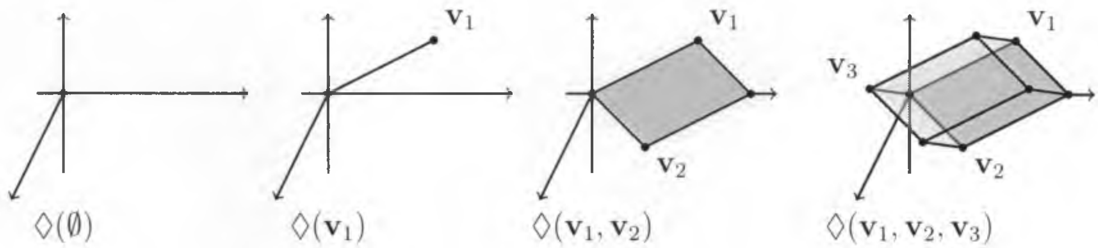


Figure 2.5: Parallelepipeds living in $\mathbb{R}^3$ with dimension $d = 0, 1, 2, 3$ respectively.

3-polytope with three pairs of parallel faces. See Figure 2.5.

The **unit d-cube** $C^d := [0,1]^d$ forms the simplest example of a d-dimensional parallelepiped. Its generating vectors are the standard unit basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_d$:

$$C^d = \Diamond(\mathbf{e}_1, \ldots, \mathbf{e}_d) .$$

A related object is the **(lattice) zonotope**

$$\mathcal{Z}(\mathbf{u}_1, \ldots, \mathbf{u}_r) := \left\{ \sum_{i \in [r]} \lambda_i \mathbf{u}_i : 0 \le \lambda_i \le 1 \right\} ,$$

where $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are vectors in $\mathbb{Z}^m$. We call $\mathbf{u}_1, \ldots, \mathbf{u}_r$ the **generators** of $\mathcal{Z}(\mathbf{u}_1, \ldots, \mathbf{u}_r)$. When the generators are clear, we will drop the argument and simply write $\mathcal{Z}$. If the generators of $\mathcal{Z}$ are linearly independent, then $\mathcal{Z}(\mathbf{u}_1, \ldots, \mathbf{u}_r)$ is an r-dimensional parallelepiped and

$$\mathcal{Z}(\mathbf{u}_1, \ldots, \mathbf{u}_r) = \Diamond(\mathbf{u}_1, \ldots, \mathbf{u}_r) .$$

A **subdivision** $\mathscr{C}$ of a polytope $\mathcal{P}$ is a non-empty, finite collection of polytopes such that

(1) $\mathcal{Q} \in \mathscr{C}$ implies all faces of $\mathcal{Q}$ are also in $\mathscr{C}$,

(2) $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathscr{C}$ implies $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is a common face of both $\mathcal{Q}_1$ and $\mathcal{Q}_2$,

(3) $\bigcup_{\mathcal{Q} \in \mathscr{C}} \mathcal{Q} = \mathcal{P}$.

The polytopes that comprise the subdivision $\mathscr{C}$ are called **cells** of $\mathscr{C}$. Furthermore, those cells $\mathcal{Q} \in \mathscr{C}$ with $\dim(\mathcal{Q}) = \dim(\mathcal{P})$ are called **maximal** [3].

We further note that the collection of maximal cells $\mathcal{Q}_1, \mathcal{Q}_2, \ldots \mathcal{Q}_s$ in a subdivision of a polytope $\mathcal{P}$ forms a **dissection** of $\mathcal{P}$. That is, [3]

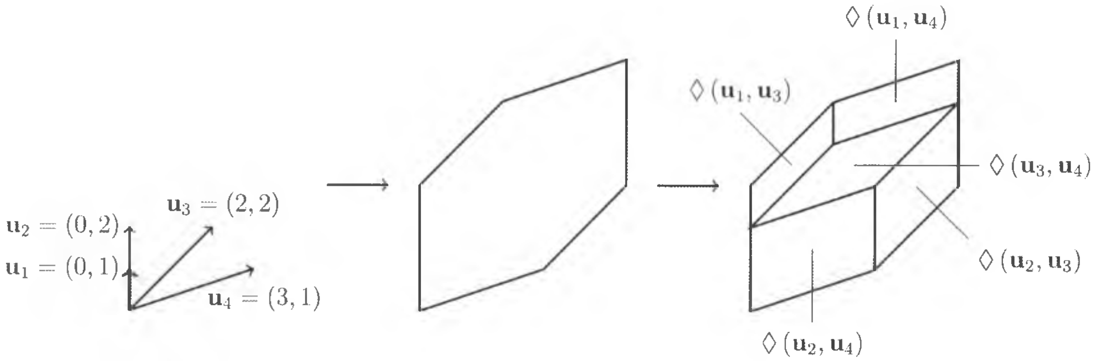
$$\mathcal{P} = \mathcal{Q}_1 \cup \mathcal{Q}_2 \cup \cdots \cup \mathcal{Q}_s$$

and

$$\mathcal{Q}_i^\circ \cap \mathcal{Q}_j^\circ = \emptyset \text{ whenever } i \neq j\,.$$

The following theorem due to Shephard makes explicit the close relationship between zonotopes and parallelepipeds.

Theorem 2.1 [11, Theorems 54 and 56]. *Every zonotope admits a subdivision into parallelepipeds. In particular, every d-dimensional zonotope $\mathcal{Z}(\mathbf{u}_1, \ldots, \mathbf{u}_r)$ admits a subdivision $\mathscr{C}$ into parallelepipeds where the maximal cells in $\mathscr{C}$ are generated by the linearly independent subsets of $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ of size d.*



(a) generators in $\mathbb{Z}^2$ (b) $\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ (c) a subdivision of $\mathcal{Z}$

Figure 2.6: Generators in $\mathbb{Z}^2$, the 2-dimensional zonotope $\mathcal{Z}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$, and a subdivision of $\mathcal{Z}$ into parallelepipeds.

Figure 2.6 shows a 2-dimensional zonotope $\mathcal{Z}$ generated by vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \in \mathbb{Z}^2$ and a subdivision $\mathscr{C}$ of $\mathcal{Z}$ into parallelepipeds. The maximal cells in $\mathscr{C}$ are labeled[3]. Observe that the generators of these maximal cells are exactly those linearly independent subsets of $\{\mathbf{u}_1, \ldots, \mathbf{u}_4\}$ of size two. Further observe how the maximal cells of $\mathscr{C}$ form a dissection of $\mathcal{Z}$.

A lattice polytope $\mathcal{P}$ is **integrally closed** if for all integers $n \geq 1$ and for all lattice points $\mathbf{p} \in n\mathcal{P}$, there exist lattice points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \in \mathcal{P}$ such that

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n \,.$$

Parallelepipeds and zonotopes are both examples of integrally closed polytopes. We

[3]We abuse notation here: each maximal cell in $\mathscr{C}$ is in fact a translate of the parallelepiped by which it is labeled.

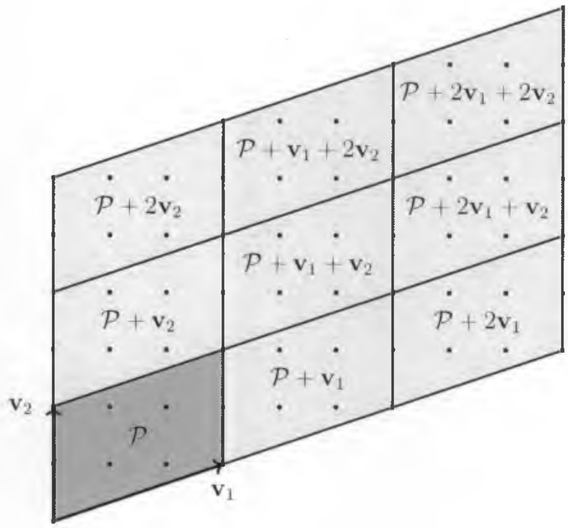


Figure 2.7: We can tile the third dilate of 2-parallelepiped $\mathcal{P}$ with 3^2 copies of $\mathcal{P}$.

use induction and a simple tiling argument to show that parallelepipeds are integrally closed. A similar argument holds for zonotopes as a result of Theorem 2.1.

Let $\mathcal{P} = \lozenge(\mathbf{v}_1, \ldots, \mathbf{v}_d) \subset \mathbb{R}^m$ be a lattice d-parallelepiped. When $n = 1$, the desired result is trivially true. Let $n \geq 2$. We can tile $n\mathcal{P}$ with $n^{\dim(\mathcal{P})} = n^d$ copies of $\mathcal{P}$:

$$n\mathcal{P} = \bigcup (\mathcal{P} + \mathbf{v}) ,$$

where the union runs over all linear combinations of $\mathbf{v} = k_1\mathbf{v}_1 + \cdots + k_d\mathbf{v}_d$ with $k_i \in \{0, 1, \ldots, n-1\}$. See Figure 2.7 for an example with $n = 3$ and $d = 2$. Choose $\mathbf{p} \in n\mathcal{P} \cap \mathbb{Z}^m$ such that $\mathbf{p} \notin (n-1)\mathcal{P} \cap \mathbb{Z}^m$. Then $\mathbf{p} \in (\mathcal{P} \cap \mathbb{Z}^m) + \mathbf{v}$ for some $\mathbf{v} = k_1\mathbf{v}_1 + \cdots + k_d\mathbf{v}_d$ with at least one k_i equal to $n - 1$. If this is not the case, then we have $k_i \in \{0, 1, \ldots, n-2\}$ for all $i \in [d]$. But this implies $\mathbf{p} \in (n-1)\mathcal{P} \cap \mathbb{Z}^m$, a contradiction.

We know $\mathbf{v} \in (n-1)\mathcal{P} \cap \mathbb{Z}^m$ by definition. Therefore,

$$\mathbf{v} = \mathbf{p}_1 + \cdots + \mathbf{p}_{n-1}$$

for some $\mathbf{p}_1, \ldots, \mathbf{p}_{n-1} \in \mathcal{P} \cap \mathbb{Z}^m$ by the induction hypothesis. Furthermore, $\mathbf{p} \in (\mathcal{P} \cap \mathbb{Z}^m) + \mathbf{v}$ implies $\mathbf{p} = \mathbf{p}_0 + \mathbf{v}$ for some $\mathbf{p}_0 \in \mathcal{P} \cap \mathbb{Z}^m$. Thus we have

$$\mathbf{p} = \mathbf{p}_0 + \mathbf{p}_1 + \cdots + \mathbf{p}_{n-1},$$

concluding our argument that parallelepipeds are integrally closed.

Chapter 3

Half-Open Parallelepipeds and the (A, j)-Eulerian Numbers

3.1 Descent Statistics

Let S_d denote the set of all permutations on $[d] := \{1, \ldots, d\}$. We use permutation words $\sigma = \sigma_1\sigma_2 \ldots \sigma_d$ in single line notation to denote permutations in S_d. For example, the permutation word $\sigma = 4213$ in S_4 is equivalent to the permutation seen in two-line notation here:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{bmatrix}.$$

We call i a **descent** of σ if $\sigma_i > \sigma_{i+1}$. For example, the descents of $\sigma = 4213$ are 1 and 2. Observe these descents in a permutation diagram of $\sigma = 4213$ in Figure 3.5.

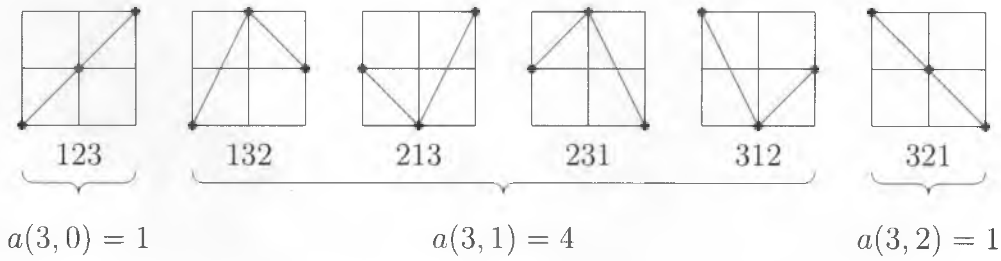


Figure 3.1: Permutation diagrams of S_3 with descents in red, grouped by descent number.

The collection of descents of a permutation σ is called the **descent set** of σ. The cardinality of the descent set is called the **descent number** of σ. We write

$$\begin{aligned}\mathrm{Des}(\sigma) &:= \{i \in [d-1] : \sigma_i > \sigma_{i+1}\} \quad \text{and} \\ \mathrm{des}(\sigma) &:= |\,\mathrm{Des}(\sigma)|\end{aligned}$$

for the descent set and the descent number of σ, respectively. We similarly define the **ascent set** and the **ascent number** of a permutation, respectively, as follows:

$$\begin{aligned}\mathrm{Asc}(\sigma) &:= \{i \in [d-1] : \sigma_i < \sigma_{i+1}\} \quad \text{and} \\ \mathrm{asc}(\sigma) &:= |\mathrm{Asc}(\sigma)|\ .\end{aligned}$$

We note that d is neither a descent nor an ascent of $\sigma \in S_d$. It follows that $0 \leq \mathrm{des}(\sigma), \mathrm{asc}(\sigma) \leq d-1$.

We define the **(type-A) Eulerian number** $a(d,k)$ to be the number of permutations in S_d with exactly k descents. That is,

$$a(d,k) = |\{\sigma \in S_d : \mathrm{des}(\sigma) = k\}|.$$

See Figure 3.1 for the Eulerian numbers when $d = 3$. We further define the **(type-A) Eulerian polynomial** $A(d,t)$ to be the degree-$(d-1)$ polynomial whose k^{th} term has coefficient $a(d,k)$:

$$A(d,t) := \sum_{k=0}^{d-1} a(d,k)t^k.$$

From Figure 3.1 we see that when $d = 3$ we have the Eulerian polynomial $A(3,t) = 1 + 4t + t^2$.

Eulerian numbers play an important role in the Ehrhart theory of parallelepipeds. For example, the Eulerian polynomial is the δ-polynomial of the unit cube. That is,

$$\delta(C^d, t) = A(d,t). \tag{3.1}$$

See [2] for example, where Eulerian numbers are in fact defined via this equality. As we will see, variants of the Eulerian numbers also appear in the δ-vector for other families of parallelepipeds. Let us motivate the results described in later sections and the methods used to prove them by sketching a proof of (3.1).

The main idea behind our proof of (3.1) is the decomposition of C^d into disjoint half-open lattice polytopes, $\mathcal{P}_1, \ldots, \mathcal{P}_r$, whose δ-polynomials we already know. If

$$C^d = \bigsqcup_{i \in [r]} \mathcal{P}_i \,,$$

where the $\mathcal{P}_i$ are lattice d-polytopes, then

$$\mathsf{Ehr}(C^d, t) = \sum_{i \in [r]} \mathsf{Ehr}(\mathcal{P}_i, t) = \sum_{i \in [r]} \frac{\delta(\mathcal{P}_i, t)}{(1-t)^{d+1}} = \frac{\sum_{i \in [r]} \delta(\mathcal{P}_i, t)}{(1-t)^{d+1}} \,,$$

which implies

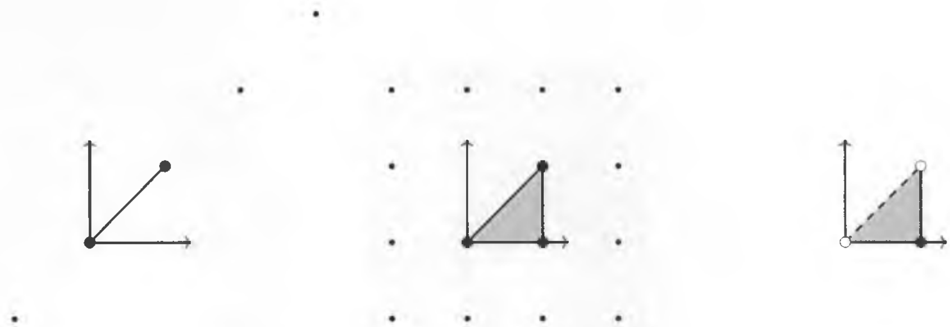
$$\delta(C^d, t) = \sum_{i \in [r]} \delta(\mathcal{P}_i, t) \,. \tag{3.2}$$

The δ-vector of a polytope encodes information about the lattice point count of that polytope. Thus it is necessary that the $\mathcal{P}_i$ in the decomposition are disjoint. If the $\mathcal{P}_i$ are not disjoint and they share any lattice points, then the sum of the lattice points in the union will not reflect the lattice point count of C^d.

To obtain the disjoint union we desire, we decompose C^d into half-open, unimodular simplices. A **simplex** $\triangle$ is a d-dimensional polytope with $d+1$ vertices. As the notation suggests, simplices are higher-dimensional analogues of triangles. A d-simplex $\triangle = \mathrm{conv}\,\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d\}$ is **unimodular** if

$$\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_d - \mathbf{v}_0$$

form a basis for the sub-lattice $\mathbb{Z}^m \cap \mathrm{span}(\triangle)$ [14]. In Figure 3.2, $\triangle_a$ and $\triangle_b$



$\triangle_a = \operatorname{conv}\{(0,0),(1,1)\}$ $\triangle_b = \operatorname{conv}\{(0,0),(1,1),(1,0)\}$ $\triangle_c = \triangle_b \setminus \triangle_a$

Figure 3.2: A unimodular 1-simplex, a unimodular 2-simplex and a half-open unimodular 2-simplex.

are unimodular simplices of dimension 1 and 2 respectively. This is best seen by computing the set $\{\mathbf{v}_j - \mathbf{v}_0 : j \in [d]\}$ for each simplex, where $\mathbf{v}_0$ is the vertex of your choice. The span of each simplex is represented in yellow. For $\triangle_a$, the intersection $\operatorname{span}(\triangle_a) \cap \mathbb{Z}^2$ is isomorphic to $\mathbb{Z}$, whereas $\operatorname{span}(\triangle_b) \cap \mathbb{Z}^2$ is all of $\mathbb{Z}^2$. We say a polytope $\mathcal{P}$ is **half-open** if one or more of its facets have been removed. In Figure 3.2, $\triangle_c$ is a half-open simplex with exactly one facet removed.

A lattice simplex $\triangle$ is unimodular if and only if it has normalized volume 1 [14]. Combining this with (2.1) and (2.2), we learn that $\delta(\triangle, t) = 1$ if and only if $\triangle$ is unimodular. Therefore, the Ehrhart series of the unimodular simplices $\triangle_a$ and $\triangle_b$ are

$$\frac{1}{(1-t)^2} \quad \text{and} \quad \frac{1}{(1-t)^3},$$

respectively, and thus the Ehrhart series of the half-open unimodular simplex $\triangle_c$ is

$$\frac{1}{(1-t)^3} - \frac{1}{(1-t)^2} = \frac{t}{(1-t)^3}\,.$$

More generally, the Ehrhart series of a d-dimensional half-open unimodular simplex with k facets removed is

$$\frac{t^k}{(1-t)^{d+1}}\,.$$

See [14], for example.

Let $\sigma \in S_d$ and define

$$\triangle_\sigma^d := \left\{ \mathbf{x} \in C^d : x_{\sigma_1} \leq x_{\sigma_2} \leq \cdots \leq x_{\sigma_d} \text{ with } x_{\sigma_i} < x_{\sigma_{i+1}} \text{ when } i \in \mathrm{Des}(\sigma) \right\}\,.$$

A strict inequality in $\triangle_\sigma^d$ corresponds to the removal of one facet. So $\triangle_\sigma^d$ is a half-open, unimodular simplex with exactly $\mathrm{des}(\sigma)$ facets removed. Every point $\mathbf{x}$ in C^d is contained in $\triangle_\sigma^d$ for some permutation σ. Furthermore, the strict inequalities at the descents ensure that the union is disjoint [1, 14]. Thus

$$C^d = \bigsqcup_{\sigma \in S_d} \triangle_\sigma^d\,.$$

We see this in Figure 3.3 for the 2-cube. We can now conclude with (3.2) that

$$\delta(C^d, t) = \sum_{\sigma \in S_d} \delta(\triangle_\sigma^d, t) = \sum_{\sigma \in S_d} t^{\mathrm{des}(\sigma)} = \sum_{k=0}^{d-1} a(d,k) t^k = A(d,t)\,.$$

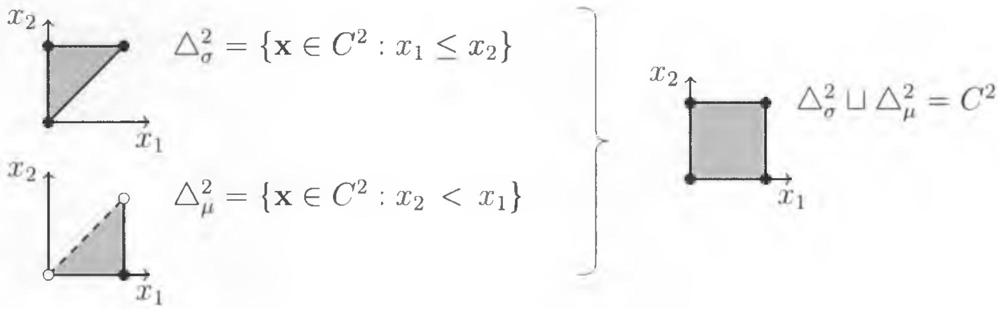


Figure 3.3: The decomposition of C^2 into disjoint half-open unimodular simplices.

That is, the δ-polynomial of C^d is the Eulerian polynomial, and this proves (3.1).

It is well known that the Eulerian polynomial is palindromic. Equivalently we say that $A(d,t)$ is **symmetric** with **center of symmetry** at $\frac{d-1}{2}$ and write

$$a(d,k) = a(d, d-1-k)$$

for all $d \geq 1$ and $0 \leq k \leq d-1$. This symmetry is easily seen via a bijection between permutations in S_d with k descents and permutations in S_d with $d-1-k$ descents:

$$\begin{aligned} \psi : S_d &\to S_d \\ \sigma &\mapsto \sigma^{\mathrm{rev}} := \sigma_d \sigma_{d-1} \cdots \sigma_1 \,. \end{aligned}$$

Observe the symmetry via permutation diagrams of $\sigma = 4213$ and $\sigma^{\mathrm{rev}} = 3124$ in Figure 3.5.

In addition to being palindromic, the Eulerian polynomial is also unimodal (see

Theorem 3.3 below). Together these imply that $A(d,t)$ is alternatingly increasing. More generally, the symmetry and unimodality of a polynomial together imply that it must peak at the coefficient(s) closest to the center of symmetry. Therefore, when d is odd, $A(d,t)$ has a single peak at $k = \frac{d-1}{2}$, and when d is even, $A(d,t)$ has a double peak at $k = \lfloor \frac{d-1}{2} \rfloor$ and $k = \lfloor \frac{d-1}{2} \rfloor + 1$. (This is confirmed by Theorem 3.3.)

3.2 Half-Open Unit Cubes

We begin this section by introducing another descent statistic and a refinement of the Eulerian numbers. For $d \geq 1$, $1 \leq j \leq d$, and $k \in \mathbb{Z}$ we define the (A, j)**-Eulerian number**

$$a_j(d,k) := |\{\sigma \in S_d \colon \sigma_d = d+1-j \text{ and } \operatorname{des}(\sigma) = k\}|$$

and the (A, j)**-Eulerian polynomial**

$$A_j(d,t) := \sum_{k=0}^{d-1} a_j(d,k) t^k \, .$$

For all $k < 0$ and for all $k \geq d$ we have $a_j(d,k) = 0$. We treat $A_j(d,t)$ as having degree $d-1$, though its leading term may be zero. See Figure 3.4 for the (A,j)-Eulerian polynomials for $d = 3$.

The following lemma is due to [5]. We elaborate on the details of their argument in the proof below which gives a bijection between permutations $\sigma \in S_d$ with k

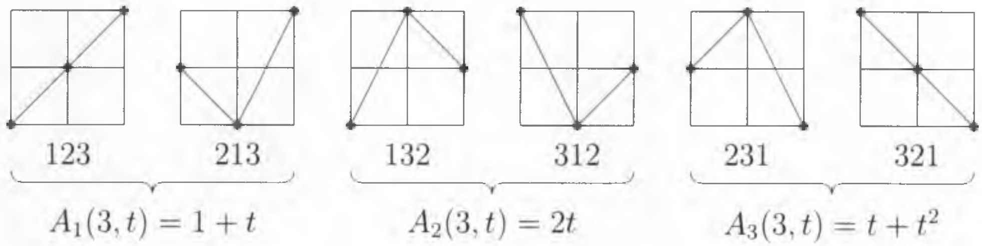


Figure 3.4: The permutation diagrams of S_3 grouped by last letter reveal the (A, j)-Eulerian polynomials for $d = 3$.

descents and last letter $d+1-j$ and permutations $\sigma^{\text{flip}} \in S_d$ with $d-1-k$ descents and last letter j. The result appears later in [10, Lemma 2.3], though in this case the coefficients of $A_j(d,t)$ are not defined in terms of descent statistics on S_d.

Lemma 3.1 [5, Lemma 2.5]. *For all $d \geq 1$, $1 \leq j \leq d$ and $k \in \mathbb{Z}$,*

$$a_j(d,k) = a_{d+1-j}(d, d-1-k)\,, \text{ equivalently}$$
$$A_j(d,t) = t^{d-1} A_{d+1-j}\left(d, \frac{1}{t}\right).$$

Proof. Fix $d \geq 1$. Consider the bijection

$$\psi : S_d \to S_d$$
$$\sigma \mapsto \sigma^{\text{flip}}\,,$$

where $\sigma_i^{\text{flip}} := d + 1 - \sigma_i$ (for a diagram see Figure 3.5). Then

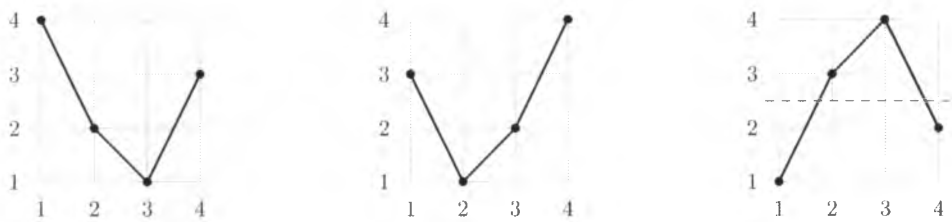
$$\begin{aligned} i \in \operatorname{Des}(\sigma) \quad &\iff \quad \sigma_i > \sigma_{i+1} \\ &\iff \quad d + 1 - \sigma_i < d + 1 - \sigma_{i+1} \\ &\iff \quad \sigma_i^{\text{flip}} < \sigma_{i+1}^{\text{flip}} \\ &\iff \quad i \in \operatorname{Asc}\left(\sigma^{\text{flip}}\right) . \end{aligned}$$

Therefore,

$$\operatorname{des}(\sigma) = k \quad \iff \quad \operatorname{asc}\left(\sigma^{\text{flip}}\right) = k \quad \iff \quad \operatorname{des}\left(\sigma^{\text{flip}}\right) = d - 1 - k .$$

Also observe $\sigma_d = d + 1 - j$ if and only if $\sigma_d^{\text{flip}} = j$. We conclude

$$a_j(d, k) = a_{d+1-j}(d, d - 1 - k) .$$



$\sigma = 4213$ $\qquad$ $\sigma^{\text{rev}} = 3124$ $\qquad$ $\sigma^{\text{flip}} = 1342$

Figure 3.5: Permutation diagrams of σ, σ^{rev} and σ^{flip} respectively.

This gives us the following:

$$\begin{aligned}
t^{d-1} A_{d+1-j}\left(d, \frac{1}{t}\right) &= t^{d-1} \sum_{k=0}^{d-1} a_{d+1-j}(d,k) t^{-k} \\
&= \sum_{k=0}^{d-1} a_{d+1-j}(d,k) t^{d-1-k} \\
&= \sum_{k=0}^{d-1} a_{d+1-j}(d, d-1-k) t^{k} \\
&= \sum_{k=0}^{d-1} a_{j}(d,k) t^{k} \\
&= A_j(d,t)\,. \qquad \square
\end{aligned}$$

The following lemma gives a recursive formula for the (A, j)-Eulerian numbers and polynomials. The result is due to [5]. Once again, we provide an elaboration of their argument.

Lemma 3.2 [5, Lemma 2.5].

$$a_j(d+1,k) = \sum_{l=1}^{j-1} a_l(d,k-1) + \sum_{l=j}^{d} a_l(d,k)\,, \quad \textit{equivalently}$$

$$A_j(d+1,t) = t\sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t)\,.$$

Proof. Fix $d \geq 1$, $1 \leq j \leq d$ and $k \in \mathbb{Z}$. Let $\sigma \in S_{d+1}$ with k descents and last letter $d+2-j$. Furthermore, let $\mu = \sigma_1\sigma_2\ldots\sigma_d$ and $K = [d+1] \setminus \{d+2-j\}$. Then $\mu \in S_K$, the set of all permutations on the set K. We note that S_K is isomorphic to $S_{|K|}$ or S_d.

Case 1: $d \in \mathrm{Des}(\sigma)$.

$$d \in \mathrm{Des}(\sigma) \iff \sigma_d > \sigma_{d+1} \iff \sigma_d > d+2-j \iff d+3-j \leq \sigma_d \leq d+1\,.$$

Because σ has k descents and $d \in \mathrm{Des}(\sigma)$, μ must have exactly $k-1$ descents. Then

$$\begin{aligned}
&|\{\sigma \in S_{d+1} : d \in \mathrm{Des}(\sigma) \text{ and } \sigma_d = d+2-j \text{ and } \mathrm{des}(\sigma) = k\}| = \\
&|\{\mu \in S_K : d+3-j \leq \mu_d \leq d+1 \text{ and } \mathrm{des}(\mu) = k-1\}| = \\
&|\{\mu' \in S_d : d+2-j \leq \mu'_d \leq d \text{ and } \mathrm{des}(\mu') = k-1\}| = \\
&\sum_{l=1}^{j-1} a_l(d,k-1)\,.
\end{aligned}$$

Case 2: $d \notin \mathrm{Des}(\sigma)$.

$$d \notin \mathrm{Des}(\sigma) \iff \sigma_d < \sigma_{d+1} \iff \sigma_d < d+2-j \iff 1 \leq \sigma_d \leq d+1-j\,.$$

Furthermore, $\mathrm{des}(\sigma) = k$ and $d \notin \mathrm{Des}(\sigma)$ implies μ has k descents. This implies

$$\begin{aligned} |\{\sigma \in S_{d+1} : d \notin \mathrm{Des}(\sigma) \text{ and } \sigma_d = d+2-j \text{ and } \mathrm{des}(\sigma) = k-1\}| &= \\ |\{\mu \in S_K : 1 \leq \mu_d \leq d+1-j \text{ and } \mathrm{des}(\mu) = k\}| &= \\ \sum_{l=j}^{d} a_l(d,k)\,. \end{aligned}$$

We conclude

$$a_j(d+1,k) = \sum_{l=1}^{j-1} a_l(d,k-1) + \sum_{l=j}^{d} a_l(d,k)\,,$$

from which it follows that

$$
\begin{aligned}
A_j(d+1,t) &= \sum_{k=0}^{d} a_j(d+1,k)t^k \\
&= \sum_{k=0}^{d}\left[\sum_{l=1}^{j-1} a_l(d,k-1) + \sum_{l=j}^{d} a_l(d,k)\right] t^k \\
&= \sum_{k=0}^{d}\sum_{l=1}^{j-1} a_l(d,k-1)t^k + \sum_{k=0}^{d}\sum_{l=j}^{d} a_l(d,k)t^k \\
&= \sum_{l=1}^{j-1}\left[\sum_{k=0}^{d} a_l(d,k-1)t^k\right] + \sum_{l=j}^{d}\left[\sum_{k=0}^{d} a_l(d,k)t^k\right] \\
&= \sum_{l=1}^{j-1}\left[\sum_{k=0}^{d-1} a_l(d,k)t^{k+1}\right] + \sum_{l=j}^{d}\left[\sum_{k=0}^{d} a_l(d,k)t^k\right] \\
&= t\sum_{l=1}^{j-1}\left[\sum_{k=0}^{d-1} a_l(d,k)t^k\right] + \sum_{l=j}^{d}\left[\sum_{k=0}^{d} a_l(d,k)t^k\right] \\
&= t\sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t)\,. \qquad \square
\end{aligned}
$$

The following theorem is due to Jochemko. It can be seen as a reincarnation of the unimodality results for closed parallelepipeds by Schepers and Van Langenhoven in [10]. The proof is taken from [8].

Theorem 3.3 [8, Theorem 3.2.2]. *For all $d \geq 1$, $1 \leq j \leq d$, the coefficients of*

$A_j(d,t)$ are unimodal. More specifically, in the case that d is even we have

$$\begin{array}{lllllllll} a_j(d,0) & \leq & \dots & \leq & a_j(d,\frac{d}{2}-1) & \geq & \dots & \geq & a_j(d,d-1) \quad \text{if } 1 \leq j \leq \frac{d}{2}, \\ a_j(d,0) & \leq & \dots & \leq & a_j(d,\frac{d}{2}) & \geq & \dots & \geq & a_j(d,d-1) \quad \text{if } \frac{d}{2} < j \leq d, \end{array}$$

and if $d \geq 3$ is odd we have

$$\begin{array}{lllllllllll} a_1(d,0) & \leq & \dots & \leq & a_1(d,\lfloor\frac{d}{2}\rfloor-1) & = & a_1(d,\lfloor\frac{d}{2}\rfloor) & \geq & \dots & \geq & a_1(d,d-1,) \\ a_d(d,0) & \leq & \dots & \leq & a_d(d,\lfloor\frac{d}{2}\rfloor) & = & a_d(d,\lfloor\frac{d}{2}\rfloor+1) & \geq & \dots & \geq & a_d(d,d-1), \end{array}$$

$$a_j(d,0) \leq \dots \leq a_j(d,\lfloor\tfrac{d}{2}\rfloor) \geq \dots \geq a_j(d,d-1) \quad \text{if } 2 \leq j \leq d-1 \,.$$

Proof. When $d = 1$, the result is trivially true. For $d \geq 2$, we argue by induction on d. The case $d = 2$ is easily checked. For $d = 3$, we see from Figure 3.4 that the result holds.

Let $d+1$ be even. We then distinguish two cases:

Case: $1 \leq j < \frac{d+1}{2}$. Then

$$A_j(d+1,t) = t\sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d+1-j} A_l(d,t) + \sum_{l=d+2-j}^{d} A_l(d,t)$$

by Lemma 3.2. The first and the third summand added give by Lemma 3.1 a palindromic polynomial with center of symmetry at $\frac{d}{2}$ which, by induction, has unimodal coefficients with peaks at $\lfloor\frac{d}{2}\rfloor$ and $\lfloor\frac{d}{2}\rfloor+1$. The second summand has by induction unimodal coefficients with peak at $\lfloor \frac{d}{2} \rfloor = \frac{d+1}{2} - 1$.

Case: $\frac{d+1}{2} < j \leq d+1$. Then

$$A_j(d+1,t) = t \sum_{l=1}^{d+1-j} A_l(d,t) + t \sum_{l=d+2-j}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t)\,.$$

The first and the third summand added give a palindromic polynomial with center of symmetry at $\frac{d}{2}$, which has unimodal coefficients with peaks at $\lfloor \frac{d}{2} \rfloor$ and $\lfloor \frac{d}{2} \rfloor + 1$. In this case, the coefficients of the second summand form a unimodal sequence with peak at $\lfloor \frac{d}{2} \rfloor + 1 = \frac{d+1}{2}$.

If $d+1$ is odd, we distinguish again two cases:

Case: $1 \leq j \leq \frac{d+1}{2}$. By Lemma 3.2 we have

$$A_j(d+1,t) = t \sum_{l=1}^{j-1} A_l(d,t) + \sum_{l=j}^{d+1-j} A_l(d,t) + \sum_{l=d+2-j}^{d} A_l(d,t)\,.$$

The second summand is by induction and Lemma 3.1 a palindromic polynomial with unimodal coefficients and peaks at $\frac{d}{2} - 1$ and $\frac{d}{2}$. The coefficients of the first and third summand are unimodal with peak at $\frac{d}{2} = \lfloor \frac{d+1}{2} \rfloor$.

Case: $\frac{d+1}{2} < j \leq d+1$. We have

$$A_j(d+1,t) = t \sum_{l=1}^{d+1-j} A_l(d,t) + t \sum_{l=d+2-j}^{j-1} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t)\,.$$

As in the previous case, the coefficients of the summand in the middle are unimodal and palindromic, this time with peaks at $\frac{d}{2}$ and $\frac{d}{2}+1$. The coefficients of the first and third summand form again a unimodal sequence with peak at $\frac{d}{2} = \lfloor \frac{d+1}{2} \rfloor$. $\square$

Theorem 3.3 proves that the coefficients of the (A, j)-Eulerian polynomials are unimodal and specifies the peak(s). From the proof of Proposition 2.17 in [10], we also know that the coefficients of these polynomials are alternatingly increasing for sufficiently large j. We formally record this result and provide a proof, an extension of the proof of Theorem 3.3, below.

Lemma 3.4. *For all $d \geq 0$ and $\lfloor \frac{d+1}{2} \rfloor < j \leq d+1$, the coefficients of $A_j(d+1,t)$ are alternatingly increasing.*

Proof. Fix $d \geq 0$ and $\lfloor \frac{d+1}{2} \rfloor < j \leq d+1$. Let

$$b(t) = t \sum_{l=1}^{d+1-j} A_l(d,t) + \sum_{l=j}^{d} A_l(d,t)$$

and

$$c(t) = \sum_{l=d+2-j}^{j-1} A_l(d,t)\,.$$

Then by Lemma 3.2

$$A_j(d+1,t) = b(t) + t\,c(t) \quad \text{and} \quad a_j(d+1,k) = b_k + c_{k-1}\,,$$

where b_i and c_i are the coefficient of t^i in $b(t)$ and $c(t)$ respectively, and $c_{-1} := 0$. We wish to show that the coefficients of $A_j(d+1,t)$ are alternatingly increasing (for our given values of j). Equivalently, we will show that

$$a_j(d+1,k) \le a_j(d+1,d-k) \quad \text{for} \quad 0 \le k \le \left\lfloor \tfrac{d}{2} \right\rfloor , \tag{3.3}$$

and

$$a_j(d+1,d-k) \le a_j(d+1,k+1) \quad \text{for} \quad 0 \le k \le \left\lfloor \tfrac{d-1}{2} \right\rfloor . \tag{3.4}$$

By Lemma 3.1 and Lemma 3.3 we know $b(t)$ is symmetric and unimodal with peak(s) of unimodality in agreement with center of symmetry at $\frac{d}{2}$; and $c(t)$ is symmetric and unimodal with peak(s) of unimodality in agreement with center of symmetry at $\frac{d-1}{2}$.

Fix $0 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$. Then by the symmetry and unimodality of $b(t)$ and $c(t)$,

$$a_j(d+1,k) \;=\; b_k + c_{k-1} \;=\; b_{d-k} + c_{d-k} \;\le\; b_{d-k} + c_{d-k-1} \;=\; a_j(d+1,d-k)\,.$$

This establishes (3.3).

Now fix $0 \le k \le \left\lfloor \frac{d-1}{2} \right\rfloor$. Then

$$a_j(d+1,d-k) \;=\; b_{d-k} + c_{d-k-1} \;=\; b_k + c_k \;\le\; b_{k+1} + c_k \;=\; a_j(d+1,k+1)\,,$$

also by the symmetry and unimodality of $b(t)$ and $c(t)$. This establishes (3.4). We conclude that $A_j(d+1,t)$ is alternatingly increasing for $\left\lfloor \frac{d+1}{2} \right\rfloor < j \le d+1$. $\square$

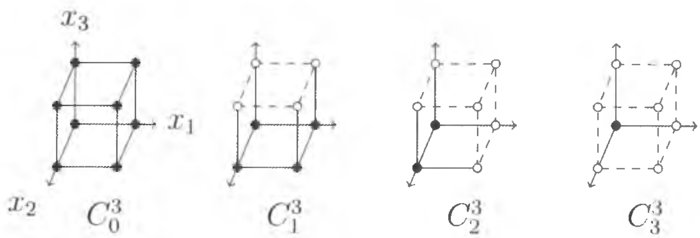


Figure 3.6: Half-open unit cubes C_j^3 for $j = 0, 1, 2, 3$.

For $j \in \{0, \dots, d\}$ we define the **half-open unit cube**

$$C_j^d := [0,1]^d \setminus \{x_d = 1, x_{d-1} = 1, \dots, x_{d+1-j} = 1\} .$$

The subscript j indicates the number of facets removed from C^d to obtain C_j^d. We note that for $j > 0$, the intersection of the removed facets of C_j^d is non-empty.

We define the **j-descent set** $\mathrm{Des}_j(\sigma) \subseteq \{1, \dots, d\}$ of a permutation $\sigma \in S_d$ by

$$\mathrm{Des}_j(\sigma) := \begin{cases} \mathrm{Des}(\sigma) \cup \{d\} & \text{if } d+1-j \leq \sigma_d \leq d\,, \\ \mathrm{Des}(\sigma) & \text{otherwise}\,. \end{cases}$$

Further, we define the **j-descent number** $\mathrm{des}_j(\sigma)$ to be the cardinality of the j-descent set.

We can now describe the (A, j)-Eulerian numbers in terms of j-descents, as the following lemma shows us. The result is due to Jochemko and appears in [8].

Lemma 3.5 [8, Lemma 3.3.1]. *Let $\sigma \in S_d$. Then for all $0 \leq j \leq d$ and $0 \leq k \leq d$*

$$|\{\sigma \in S_d : \mathrm{des}_j(\sigma) = k\}| = a_{j+1}(d+1, k)\,.$$

Proof. Fix $0 \leq j \leq d$ and $0 \leq k \leq d$. Consider $\sigma \in S_d$ with $\mathrm{des}_j(\sigma) = k$. By definition

$$\mathrm{des}_j(\sigma) = \begin{cases} \mathrm{des}(\sigma) + 1 & \text{if } d+1-j \leq \sigma_d \leq d\,, \\ \mathrm{des}(\sigma) & \text{otherwise}\,. \end{cases}$$

If $d+1-j \leq \sigma_d \leq d$, then $\mathrm{des}_j(\sigma) = k$ implies that $\mathrm{des}(\sigma) = k-1$. If $1 \leq \sigma_d \leq d-j$, then $\mathrm{des}_j(\sigma) = k$ implies that $\mathrm{des}(\sigma) = k$. So the number of permutations in S_d with j-descent number k is equal to the number of permutations in S_d with descent number $k-1$ and last letter between $d+1-j$ and d (inclusive) *plus* the number of permutations in S_d with descent number k and last letter strictly less than $d+1-j$. Equivalently,

$$\begin{aligned} |\{\sigma \in S_d\colon \mathrm{des}_j(\sigma) = k\}| &= \sum_{l=1}^{j} a_l(d, k-1) + \sum_{l=j+1}^{d} a_l(d,k) \\ &= a_{j+1}(d+1, k)\,, \end{aligned}$$

where the last equality follows from Lemma 3.2. $\square$

Remark. Lemma 3.5 also holds as a result of the proof of equations (4.3) and (4.4). The bijection we use in the proof is a modified version of that used by Jochemko in [8].

The following result comes from [8]; it is a generalization of property (1) of Lemma 2.5 in [10]. We provide an altered version of Jochemko's proof below.

Proposition 3.6 [8, Theorem 3.3.2]. *Let $d \geq 1$ and $0 \leq j \leq d$. Then*

$$\mathsf{Ehr}\left(C_j^d, t\right) = \frac{A_{j+1}(d+1,t)}{(1-t)^{d+1}}.$$

Proof. Let $d \geq 1$, $0 \leq j \leq d$ and $\sigma \in S_d$. Define the half-open unimodular simplex

$$\triangle_\sigma^{d,j} := \left\{ \begin{array}{c} \mathbf{x} \in C_j^d : x_{\sigma_1} \leq x_{\sigma_2} \leq \cdots \leq x_{\sigma_d} \\ \text{with } x_{\sigma_i} < x_{\sigma_{i+1}} \text{ when } i \in \mathrm{Des}(\sigma) \end{array} \right\}.$$

This gives us the disjoint union,

$$C_j^d = \bigsqcup_{\sigma \in S_d} \triangle_\sigma^{d,j}.$$

We also have the following equivalent definitions

$$\triangle_\sigma^{d,j} = \left\{ \begin{array}{c} \mathbf{x} \in \mathbb{R}^d : 0 \le x_{\sigma_1} \le x_{\sigma_2} \le \cdots \le x_{\sigma_d} \le 1 \\ \text{with } x_{\sigma_i} < x_{\sigma_{i+1}} \text{ when } i \in \mathrm{Des}(\sigma) \\ \text{and } x_{\sigma_d} < 1 \text{ when } d+1-j \le \sigma_d \le d \end{array} \right\}$$

$$= \left\{ \begin{array}{c} \mathbf{x} \in \mathbb{R}^d : 0 \le x_{\sigma_1} \le x_{\sigma_2} \le \cdots \le x_{\sigma_d} \le 1 \\ \text{with } x_{\sigma_i} < x_{\sigma_{i+1}} \text{ when } i \in \mathrm{Des}_j(\sigma) \\ \text{and } x_{\sigma_d} < 1 \text{ when } d \in \mathrm{Des}_j(\sigma) \end{array} \right\}.$$

Each strict inequality corresponds bijectively to a missing facet in the simplex. We see that $\triangle_\sigma^{d,j}$ has $\mathrm{des}(\sigma)+1$ missing facets when $d+1-j \le \sigma_d$ and $\mathrm{des}(\sigma)$ missing facets otherwise. Therefore, the half-open unimodular simplex $\triangle_\sigma^{d,j}$ has exactly $\mathrm{des}_j(\sigma)$ missing facets. Together with the disjoint union, this implies

$$\begin{aligned} \mathsf{Ehr}\left(C_j^d, t\right) &= \sum_{\sigma \in S_d} \mathsf{Ehr}\left(\triangle_\sigma^{d,j}, t\right) \\ &= \frac{\sum_{\sigma \in S_d} \delta\left(\triangle_\sigma^{d,j}, t\right)}{(1-t)^{d+1}} \\ &= \frac{\sum_{\sigma \in S_d} t^{\mathrm{des}_j(\sigma)}}{(1-t)^{d+1}} \\ &= \frac{\sum_{k=0}^{d} a_{j+1}(d+1,k)t^k}{(1-t)^{d+1}} \\ &= \frac{A_{j+1}(d+1,t)}{(1-t)^{d+1}}, \end{aligned}$$

where the second to last inequality follows from Lemma 3.5. □

This proposition tells us that the δ-polynomial for the half-open unit cube C_j^d is the (A, j)-Eulerian polynomial $A_{j+1}(d+1, t)$. Together with Theorem 3.3 and Lemma 3.4, we see that the δ-polynomial for C_j^d is unimodal for all $0 \leq j \leq d$ and alternatingly increasing for $\left\lfloor \frac{d+1}{2} \right\rfloor \leq j \leq d$.

3.3 Half-Open Parallelepipeds

Let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be linearly independent vectors in $\mathbb{Z}^d$ and let $J \subseteq [d]$. Then we define respectively the **closed parallelepiped** $\Diamond(J)$, the **standard half-open parallelepiped** $\Pi(J)$, and the **open parallelepiped** $\Box(J)$ generated by J [10]:

$$\Diamond(J) := \left\{ \sum_{j \in J} \lambda_j \mathbf{v}_j : 0 \leq \lambda_j \leq 1 \right\},$$

$$\Pi(J) := \left\{ \sum_{j \in J} \lambda_j \mathbf{v}_j : 0 \leq \lambda_j < 1 \right\},$$

$$\Box(J) := \left\{ \sum_{j \in J} \lambda_j \mathbf{v}_j : 0 < \lambda_j < 1 \right\}.$$

Let F_i be the facet of $\Diamond(J)$ generated by $J \setminus \{i\}$ and let F_i' be the translate of F_i in $\Diamond(J)$:

$$F_i := \Diamond\left(J \setminus \{i\}\right) \quad \text{and} \quad F_i' := \mathbf{v}_i + F_i\,.$$

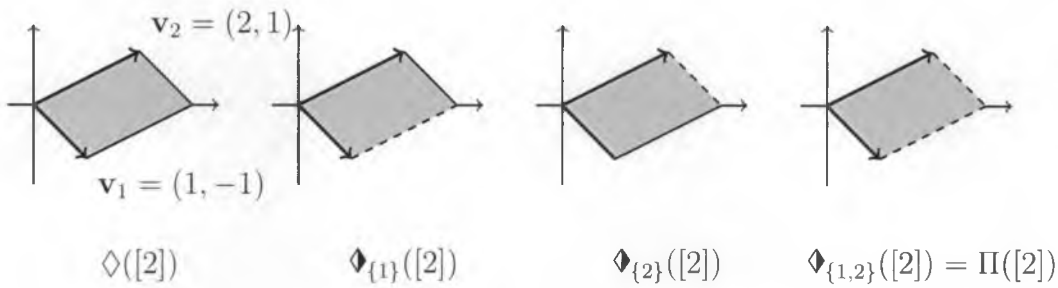


Figure 3.7: A closed 2-dimensional parallelepiped and its half-open variants.

Note that

$$\Pi(J) = \lozenge(J) \setminus \bigcup_{i \in J} F_i'$$

and

$$\square(J) = \lozenge(J) \setminus \bigcup_{i \in J} (F_i \cup F_i') \ .$$

Let $I \subseteq J \subseteq [d]$. Then we define

$$\blacklozenge_I(J) := \left\{ \sum_{j \in J} \lambda_j \mathbf{v}_j : 0 \leq \lambda_j \leq 1 \text{ with } \lambda_j < 1 \text{ for all } j \in I \right\}$$

to be the **half-open parallelepiped** generated by J with omitted facets F_i', for all $i \in I$. In Figure 3.7 we see a closed 2-dimensional parallelepiped and its half-open variants.

The lemma below is a modification of a well-known result which states that a polytope is the disjoint union of the relative interiors of its faces. That is,

$$\mathcal{P} = \bigsqcup_{\mathcal{F} \subseteq \mathcal{P}} \mathcal{F}^{\circ} .$$

The lemma also appears in a more general form in [8]. The stronger result seen there applies more generally to $\mathbb{Z}^d$-valuations[1].

Lemma 3.7 [8, Lemma 3.4.3]. *Let $J \subseteq [d]$. Then*

$$\#\left(\Pi(J) \cap \mathbb{Z}^d\right) = \sum_{\emptyset \subseteq J' \subseteq J} \#\left(\square(J') \cap \mathbb{Z}^d\right) .$$

Lemma 3.8 below is a generalization of [10, Lemma 2.1]. The original result is for closed (lattice) parallelepipeds $\Diamond\left([d]\right) = \blacklozenge_{\emptyset}\left([d]\right)$. Jochemko extends it to half-open (lattice) parallepipeds in [8, Lemma 3.4.1]. In fact, she proves a stronger statement which applies to arbitrary $\mathbb{Z}^d$-valuations. We follow the proof from [10] for closed parallelepipeds, adjusting as necessary for the half-open condition.

Lemma 3.8 [10, Lemma 2.1]. *Let $I \subseteq [d]$. Then*

$$\#\left(n \blacklozenge_I\left([d]\right) \cap \mathbb{Z}^d\right) = \sum_{J' \colon I \subseteq J' \subseteq [d]} n^{|J'|} \#\left(\Pi(J') \cap \mathbb{Z}^d\right) .$$

[1] A $\mathbb{Z}^d$-valuation is a map φ from the set of all lattice polytopes in $\mathbb{R}^d$ to an abelian group such that $\varphi(\emptyset) = 0$ and $\varphi(\mathcal{P} \cap \mathcal{Q}) = \varphi(\mathcal{P}) + \varphi(\mathcal{Q}) - \varphi(\mathcal{P} \cap \mathcal{Q})$ for all lattice polytopes $\mathcal{P}, \mathcal{Q}$ such that $\mathcal{P} \cup \mathcal{Q}$ is also a lattice polytope; additionally, a $\mathbb{Z}^d$-valuation φ is invariant under translation by vectors in $\mathbb{Z}^d$, so $\varphi(\mathcal{P} + \mathbf{v}) = \varphi(\mathcal{P})$ for all lattice polytopes $\mathcal{P}$ and all $\mathbf{v} \in \mathbb{Z}^d$ [8].

Proof. Let $I \subseteq [d]$, $\square(nJ) := n\,\square(J)$, and $k' := |\{[d] \setminus (J \cup I)\}|$. Then

$$\begin{aligned}
\#\left(n\,\Phi_I\left([d]\right) \cap \mathbb{Z}^d\right) &= \sum_{J \subseteq [d]} 2^{k'} \#\left(\square(nJ) \cap \mathbb{Z}^d\right) \\
&= \sum_{J \subseteq [d]} \sum_{J \subseteq J' \subseteq [d] \text{ and } I \subseteq J'} \#\left(\square(nJ) \cap \mathbb{Z}^d\right) \\
&= \sum_{J, J' \subseteq [d]:\, J \subseteq J' \text{ and } I \subseteq J'} \#\left(\square(nJ) \cap \mathbb{Z}^d\right) \\
&= \sum_{J' \subseteq [d]:\, I \subseteq J'} \#\left(\Pi(nJ') \cap \mathbb{Z}^d\right) \\
&= \sum_{J':\, I \subseteq J'} n^{|J'|} \#\left(\Pi(J') \cap \mathbb{Z}^d\right) .
\end{aligned}$$

The second to last equality follows from Lemma 3.7. The last equality follows because $n\,\Pi(J')$ is covered by $n^{\dim(\Pi(J'))}$ translates of $\Pi(J')$ and the dimension of $\Pi(J')$ is precisely the order of the generating set J'. $\square$

Corollary 3.9 [8, Corollary 3.4.2]. *Let $d \geq 1$, $0 \leq j \leq d$ and $n \in \mathbb{Z}_{>0}$. Then*

$$\mathsf{ehr}\left(C_j^d, n\right) = \sum_{[j] \subseteq J \subseteq [d]} n^{|J|} ,$$

where we define $[0] := \emptyset$.

Proof. Fix $d \geq 1$ and $0 \leq j \leq d$. Let $n \in \mathbb{Z}_{>0}$ and $\mathbf{v}_i = \mathbf{e}_i$ for all $i \in [d]$. Then C_j^d

is congruent to $\lozenge_{[j]}([d])$. By Lemma 3.8

$$\operatorname{ehr}\left(C_j^d, n\right) = \sum_{[j]\subseteq J\subseteq[d]} n^{|J|}\,\#\left(\Pi(J)\cap\mathbb{Z}^d\right) = \sum_{[j]\subseteq J\subseteq[d]} n^{|J|},$$

where the last equality holds because the origin is the only lattice point in the half-open unit cube $\Pi(J)$. $\square$

The following theorem and corollary are due to [8], as are their proofs. The theorem is a generalization of [10, Proposition 2.2].

Theorem 3.10 [8, Theorem 3.4.4]. *Let $I \subseteq [d]$. Then the Ehrhart series of the half-open parallelepiped $\lozenge_I([d])$ is*

$$\operatorname{Ehr}\left(\lozenge_I\left([d]\right), t\right) = \frac{\sum_{K\subseteq[d]} \#\left(\square(K)\cap\mathbb{Z}^d\right) A_{|I\cup K|+1}(d+1,t)}{(1-t)^{d+1}}.$$

Proof.

$$\begin{aligned}
\mathsf{Ehr}\left(\Diamond_I\left([d]\right),t\right) &= 1+\sum_{n\geq 1}\mathsf{ehr}\left(\Diamond_I\left([d]\right),n\right)t^n \text{ by definition}\\
&= \sum_{n\geq 0}t^n\sum_{I\subseteq J}n^{|J|}\#\left(\Pi(J)\cap\mathbb{Z}^d\right) \text{ by Lemma 3.8}\\
&= \sum_{n\geq 0}t^n\sum_{I\subseteq J}n^{|J|}\sum_{K\subseteq J}\#\left(\square(K)\cap\mathbb{Z}^d\right) \text{ by Lemma 3.7}\\
&= \sum_{K\subseteq[d]}\#\left(\square(K)\cap\mathbb{Z}^d\right)\sum_{n\geq 0}t^n\sum_{(I\cup K)\subseteq J}n^{|J|}\\
&= \sum_{K\subseteq[d]}\#\left(\square(K)\cap\mathbb{Z}^d\right)\sum_{n\geq 0}t^n\,\mathsf{ehr}\left(C^d_{|I\cup K|},n\right) \text{ by Corollary 3.9}\\
&= \frac{\sum_{K\subseteq[d]}\#\left(\square(K)\cap\mathbb{Z}^d\right)A_{|I\cup K|+1}(d+1,t)}{(1-t)^{d+1}} \text{ by Proposition 3.6.}
\end{aligned}$$

The third to last equality holds because for a fixed subset K and positive integer n, the number of times that $\#\left(\square(K)\cap\mathbb{Z}^d\right)$ is counted in both expressions is equal to the number of supersets J of I that also contain K. $\square$

Corollary 3.11 [8, Corollary 3.4.5]. *The δ-polynomial of the d-dimensional half-open parallelepiped $\Diamond_I([d])$ is unimodal with peak at $\frac{d}{2}$ if d is even and with peak at $\frac{d-1}{2}$ or $\frac{d+1}{2}$ if d is odd.*

Proof. From Theorem 3.10 we have

$$\delta\left(\Diamond_I([d]),t\right)=\sum_{K\subseteq[d]}\#\left(\square(K)\cap\mathbb{Z}^d\right)A_{|I\cup K|+1}(d+1,t).$$

Furthermore $\#\left(\square(K)\cap\mathbb{Z}^d\right)\geq 0$ for all $K\subseteq[d]$. By Theorem 3.3, the coefficients of $A_{|I\cup K|+1}(d+1,t)$ form a unimodal sequence with peak at $\left\lfloor\frac{d+1}{2}\right\rfloor=\frac{d}{2}$ if d is even, and peak at $\frac{d+1}{2}-1=\frac{d-1}{2}$ or $\frac{d+1}{2}$ if d is odd. The same is true for the coefficients of any nonnegative linear combination of the $A_{|I\cup K|+1}(d+1,t)$ where $K\subseteq[d]$. $\square$

Theorem 3.10 also implies the unimodality of the δ-vector for lattice zonotopes. This is recorded formally in the following corollary, a weaker version of Jochemko's result from [8].

Corollary 3.12 [8, Theorem 3.5.4]. *The δ-polynomial of a d-dimensional zonotope is unimodal with peak at $\frac{d}{2}$ if d is even and with peak at $\frac{d-1}{2}$ or $\frac{d+1}{2}$ if d is odd.*

In order to prove this result we require a corollary to Theorem 2.1. The corollary is due to Jochemko and can be found in [8]. Her proof relies on the Beneath and Beyond construction of Köppe and Verdoolaege [9] (stated here in terms of visible faces) which gives rise to disjoint half-open decompositions of polytopes for an appropriately chosen reference point $\mathbf{p}\in\mathbb{R}^d$. We motivate and outline Jochemko's proof below.

Corollary 3.13 [8, Corollary 3.5.3]. *Every d-zonotope admits a disjoint decomposition into half-open d-parallelepipeds of the ⬗-type.*

We say a face $\mathcal{F}$ of a polytope $\mathcal{P}$ is **visible**[2] from $\mathbf{p}\in\mathbb{R}^d$ if

$$[\mathbf{p},\mathbf{x}]\cap\mathcal{P}=\{\mathbf{x}\}$$

[2] Alternatively, we say point $\mathbf{p}\in\mathbb{R}^d$ is **beyond** $\mathcal{F}$.

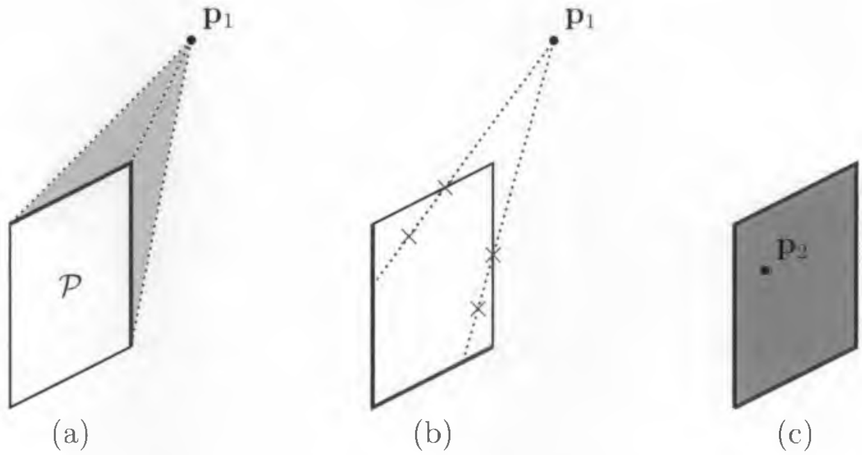


Figure 3.8: (a) All faces of $\mathcal{P}$ that are visible from $\mathbf{p}_1 \notin \mathcal{P}$ appear in blue; (b) The two facets of $\mathcal{P}$ that are not visible from $\mathbf{p}_1$ appear in red; (c) No faces of $\mathcal{P}$ are visible from $\mathbf{p}_2 \in \mathcal{P}^\circ$.

for all $\mathbf{x} \in \mathcal{F}$. In part (a) of Figure 3.8, all faces of the 2-parallelepiped $\mathcal{P}$ that are visible from $\mathbf{p}_1 \notin \mathcal{P}$ appear in blue. We note that identifying all visible facets of $\mathcal{P}$ is sufficient in identifying all visible faces of $\mathcal{P}$. This is true in general, provided the reference point lies on no facet-defining hyperplane of the polytope [3]. In part (b), the facets of $\mathcal{P}$ that are not visible from $\mathbf{p}_1$ appear in red. Observe that if facet $\mathcal{F}$ is visible from $\mathbf{p}_1$, then facet $\mathcal{F}'$, the translate of $\mathcal{F}$ in $\mathcal{P}$, is not visible from $\mathbf{p}_1$. This also holds in general [3]. In part (c), we see that no faces of $\mathcal{P}$ are visible from $\mathbf{p}_2 \in \mathcal{P}^\circ$.

From a subdivision $\mathscr{C}$ into parallelepipeds of a d-zonotope $\mathcal{Z} \subset \mathbb{R}^d$ we obtain the dissection

$$\mathcal{Z} = \lozenge_1 \cup \lozenge_2 \cup \cdots \cup \lozenge_s \,,$$

where $\Diamond_1, \Diamond_2, \ldots, \Diamond_s$ are the maximal cells of $\mathscr{C}$. This dissection, in utilizing our notion of visible faces from a fixed reference point, gives rise to a disjoint decomposition of $\mathcal{Z}$ into half-open variants of the $\Diamond_i$.

Fix $\mathbf{p} \in \mathcal{Z}^\circ$ such that $\mathbf{p}$ lies on no facet-defining hyperplane of any of the maximal cells in $\mathscr{C}$. For each $i \in [s]$, the half-open d-parallelepiped $\Diamond_i'$ is obtained by removing those faces of $\Diamond_i$ that are visible from $\mathbf{p}$:

$$\Diamond_i' := \Diamond_i \setminus \bigcup_{\mathcal{F} \text{ is visible from } \mathbf{p}} \mathcal{F}\,.$$

By construction, no two of the removed facets in $\Diamond_i'$ are translates of one another. Thus the intersection of the removed facets is empty and we rewrite $\Diamond_i'$ as $\blacklozenge_i$, as $\Diamond_i'$ is indeed a half-open parallelepiped of the $\blacklozenge$-type. This gives us the following disjoint decomposition of d-zonotope $\mathcal{Z}$ into d-parallelepipeds of the $\blacklozenge$-type:

$$\mathcal{Z} = \blacklozenge_1 \sqcup \blacklozenge_2 \sqcup \cdots \sqcup \blacklozenge_s\,.$$

See Figure 3.9 for an example in dimension 2 of how a subdivision into parallelepipeds and a reference point $\mathbf{p}$ give rise to such a decomposition.

Proof of Corollary 3.12. Let $\mathcal{Z}$ be a d-dimensional zonotope. By Corollary 3.13 there exists a disjoint decomposition of $\mathcal{Z}$ into half-open d-dimensional parallelepipeds

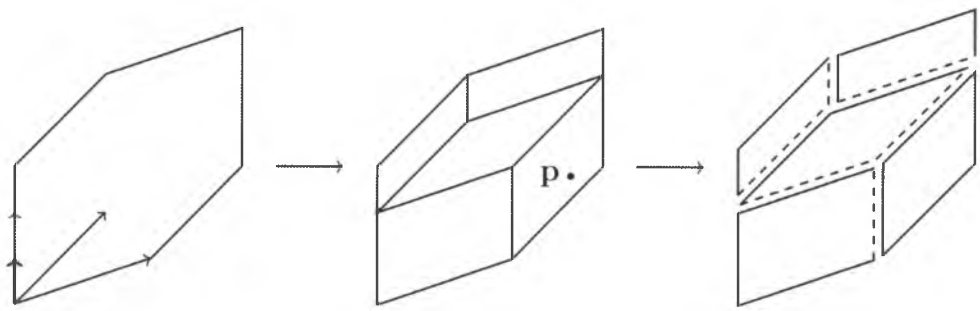


Figure 3.9: A 2-zonotope $\mathcal{Z}$, a subdivision of $\mathcal{Z}$ into parallelepipeds, and a disjoint decomposition of $\mathcal{Z}$ into half-open 2-parallelepipeds.

of the $\lozenge$-type, say

$$\mathcal{Z} = \bigsqcup_{i \in [s]} \lozenge_i \,,$$

which implies

$$\delta(\mathcal{Z}, t) = \sum_{i \in [s]} \delta(\lozenge_i, t)\,.$$

By Corollary 3.11, we know $\delta(\lozenge_i, t)$ is unimodal for all $i \in [s]$ with peak depending on the dimension of $\lozenge_i$. But all $\lozenge_i$ have the same dimension, so all $\delta(\lozenge_i, t)$ have the same peak. It follows that any sum of the δ-polynomials is unimodal with peak equal to that of each of the $\delta(\lozenge_i, t)$, and so $\delta(\mathcal{Z}, t)$ is unimodal with peak at $\frac{d}{2}$ if d is even and with peak at $\frac{d-1}{2}$ or $\frac{d+1}{2}$ if d is odd. $\square$

Chapter 4

Half-Open ± 1-Cubes and the (B, ℓ)-Eulerian Numbers

4.1 Signed Permutation Descent Statistics

A **signed permutation** on $[d]$ is a pair (π, ϵ) with $\pi \in S_d$ and $\epsilon \in \{\pm 1\}^d$. To each letter π_i in the permutation word π we assign the sign of ϵ_i, the i^{th} entry of ϵ. For a given d, the set of signed permutations is denoted by B_d and has $2^d \cdot d!$ elements. We will use one-line notation to denote signed permutation words with the following convention: those letters associated with a negative sign will be followed by an accent mark. So for $d = 5$, $\pi = 42135$ and $\epsilon = (-1, -1, 1, -1, 1)$ we write $(\pi, \epsilon) = 4'2'13'5$.

Let $\pi_0 := 0$ and $\epsilon_0 := 1$ for all $(\pi, \epsilon) \in B_d$ and all $d \geq 1$. Then $i \in [d-1] \cup \{0\}$ is a **(natural) descent** of $(\pi, \epsilon) \in B_d$ if $\epsilon_i \pi_i > \epsilon_{i+1} \pi_{i+1}$. So 0 and 3 are the descents of signed permutation word $4'2'13'5$. We easily observe the descents, seen here in red, from the two-line notation for $4'2'13'5$:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & -4 & -2 & 1 & -3 & 5 \end{bmatrix}.$$

We define the **(natural) descent set** and the **(natural) descent number** of $(\pi, \epsilon) \in B_d$, respectively, as follows:

$$\mathrm{NatDes}(\pi, \epsilon) := \left\{ i \in [d-1] \cup \{0\} : \epsilon_i \pi_i > \epsilon_{i+1} \pi_{i+1} \right\} \text{ and}$$

$$\mathrm{natdes}(\pi, \epsilon) := |\mathrm{NatDes}(\pi, \epsilon)| \ .$$

Notice that with 0 as a possible descent we have $0 \leq \mathrm{natdes}(\pi, \epsilon) \leq d$ for all $(\pi, \epsilon) \in B_d$. This differs from the descent statistic $\mathrm{des}(\sigma)$ for permutations in S_d where $0 \leq \mathrm{des}(\sigma) \leq d-1$ for all $\sigma \in S_d$.

The number of signed permutations on $[d]$ with exactly k descents is a descent statistic on B_d. We call these descent statistics the **type-B Eulerian numbers** and write

$$b(d, k) := |\{(\pi, \epsilon) \in B_d : \mathrm{natdes}(\pi, \epsilon) = k\}| \, .$$

The **type-B Eulerian polynomial** is

$$B(d, t) := \sum_{k=0}^{d} b(d, k) t^k \, .$$

Remark. The descent statistic natdes on signed permutations in B_d agrees with the descent statistic des on permutations in S_d when we fix the sign vector $\epsilon = (1, 1, \ldots, 1)$.

Consider the d-dimensional $\pm 1-$**cube**,

$$[-1, 1]^d := \left\{ \mathbf{x} \in \mathbb{R}^d : -1 \leq x_i \leq 1 \text{ for all } i \right\} ,$$

and the half-open unimodular simplex indexed by the signed permutation $(\pi, \epsilon) \in B_d$,

$$\triangle^d_{(\pi,\epsilon)} := \left\{ \begin{array}{c} \mathbf{x} \in [-1, 1]^d : 0 \leq \epsilon_1 x_{\pi_1} \leq \cdots \leq \epsilon_d x_{\pi_d} \leq 1 \\ \text{with } \epsilon_i x_{\pi_i} < \epsilon_{i+1} x_{\pi_{i+1}} \text{ when } i \in \operatorname{NatDes}(\pi, \epsilon) \end{array} \right\} .$$

The definitions imply the following disjoint union

$$[-1, 1]^d = \bigsqcup_{(\pi,\epsilon) \in B_d} \triangle^d_{(\pi,\epsilon)} ,$$

see [1]. We also observe $\triangle^d_{(\pi,\epsilon)}$ has k missing facets if and only if $\operatorname{natdes}(\pi, \epsilon) = k$.

Therefore,

$$
\begin{aligned}
\mathsf{Ehr}\left([-1,1]^d, t\right) &= \sum_{(\pi,\epsilon)\in B_d} \mathsf{Ehr}\left(\triangle^d_{(\pi,\epsilon)}, t\right) \\
&= \sum_{(\pi,\epsilon)\in B_d} \frac{\delta(\triangle^d_{(\pi,\epsilon)}, t)}{(1-t)^{d+1}} \\
&= \frac{\sum_{(\pi,\epsilon)\in B_d} t^{\mathrm{natdes}(\pi,\epsilon)}}{(1-t)^{d+1}} \\
&= \frac{B(d,t)}{(1-t)^{d+1}} .
\end{aligned}
$$

In this way we see that the Ehrhart δ-polynomial for $[-1,1]^d$ *is* the type-B Eulerian polynomial $B(d,t)$ [4, Theorem 3.4]; that is,

$$
\delta\left([-1,1]^d, t\right) = B(d,t) . \tag{4.1}
$$

The coefficient of t^k counts the half-open unimodular simplices with exactly k facets removed in the B_d-induced decomposition of $[-1,1]^d$, or equivalently, the signed permutations in B_d with exactly k descents. See Figure 4.1 for this decomposition with $d = 2$.

We wish to show that $B(d,t)$ is alternatingly increasing using the interplay of the geometry provided by the ± 1-cube and the combinatorics provided by relevant permutation descent statistics. To do this we introduce the **unit cell** indexed by

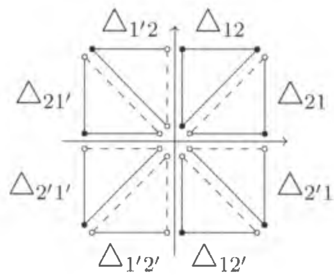


Figure 4.1: Decomposition of $[-1,1]^2$ into disjoint half-open unimodular simplices indexed by signed permutations in B_2.

$I \subseteq [d]$,

$$U_I^d := \left\{ \mathbf{x} \in [-1,1]^d : x_i \geq 0 \text{ for all } i \in I \text{ and } x_i < 0 \text{ for all } i \notin I \right\} .$$

Notice that U_I^d is a half-open unit d-cube with $d - |I|$ facets removed and that the intersection of the removed facets is non-empty, so no two of the removed facets are translates of one other. Therefore, U_I^d is congruent to C_j^d where $j = d - |I|$. We write $U_I^d \cong C_j^d$. There are 2^d unit cells indexed by the 2^d subsets of $[d]$, exactly $\binom{d}{j}$ of which are congruent to C_j^d.

The union of the unit cells is the d-dimensional ± 1-cube. Furthermore, the union is disjoint. That is,

$$[-1,1]^d = \bigsqcup_{I \subseteq [d]} U_I^d .$$

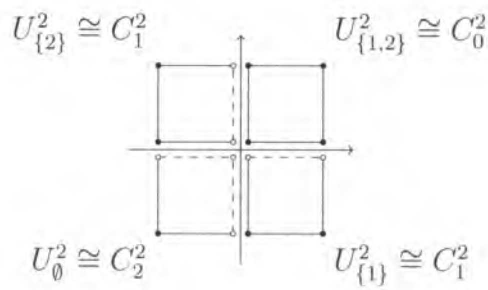


Figure 4.2: Decomposition of $[-1,1]^2$ into unit cells indexed by subsets of $\{1,2\}$.

See Figure 4.2 for this decomposition with $d = 2$. Therefore,

$$\begin{aligned}
\mathsf{Ehr}\left([-1,1^d],t\right) &= \sum_{I\subseteq[d]} \mathsf{Ehr}\left(U_I^d,t\right) \\
&= \sum_{I\subseteq[d]} \mathsf{Ehr}\left(C^d_{d-|I|},t\right) \\
&= \sum_{I\subseteq[d]} \frac{\delta\left(C^d_{d-|I|},t\right)}{(1-t)^{d+1}} \\
&= \frac{\sum_{j=0}^d \binom{d}{j}\delta\left(C^d_j,t\right)}{(1-t)^{d+1}} \\
&= \frac{\sum_{j=0}^d \binom{d}{j} A_{j+1}(d+1,t)}{(1-t)^{d+1}},
\end{aligned}$$

where the last equality follows from Proposition 3.6. From this we arrive at the following theorem and proof for the corollary.

Theorem 4.1.

$$B(d,t) = \sum_{j=0}^{d} \binom{d}{j} A_{j+1}(d+1,t) \text{ and}$$
$$b(d,k) = \sum_{j=0}^{d} \binom{d}{j} a_{j+1}(d+1,k)\,.$$

Corollary 4.2 [4, Theorem 2.4]. *The type-B Eulerian numbers are symmetric and unimodal. In particular, they are alternatingly increasing.*

Proof. In order to show that the coefficients of $B(d,t)$ are alternatingly increasing, we will consider $B(d,t)$, as in Theorem 4.1, as a positive linear combination of the (A,j)-Eulerian polynomials. We will show that this linear combination is both symmetric and unimodal, thus implying that the center of symmetry is in agreement with the peak(s) of unimodality. We will also see that the peaks are in the middle, further implying that the coefficients of $B(d,t)$ are alternatingly increasing. We proceed by cases.

Case 1: d is odd.

We rearrange $B(d,t)$ into a linear combination of polynomial pairs:

$$\begin{aligned}\sum_{j=0}^{d} \binom{d}{j} A_{j+1}(d+1,t) &= \sum_{j=0}^{\frac{d+1}{2}-1} \binom{d}{j} A_{j+1}(d+1,t) + \sum_{j=\frac{d+1}{2}}^{d} \binom{d}{d-j} A_{d+1-j}(d+1,t) \\ &= \sum_{j=0}^{\frac{d+1}{2}-1} \binom{d}{j} \left[A_{j+1}(d+1,t) + A_{d+1-j}(d+1,t)\right]\,. \qquad (4.2)\end{aligned}$$

From Lemma 3.1 we know the polynomial $A_{j+1}(d+1,t)+A_{d+1-j}(d+1,t)$ is symmetric with center of symmetry at $\frac{d}{2} = \left\lfloor \frac{d+1}{2} \right\rfloor$. Furthermore, from Theorem 3.3 we know that for all $d \geq 1$ odd, the polynomial $A_{j+1}(d+1,t)$ is unimodal with peak at

$$k = \begin{cases} \frac{d+1}{2} - 1 & \text{if } 1 \leq j+1 \leq \frac{d+1}{2} \quad \Longleftrightarrow \quad 0 \leq j \leq \frac{d+1}{2} - 1\,, \\ \frac{d+1}{2} & \text{if } \frac{d+1}{2} < j+1 \leq d+1 \quad \Longleftrightarrow \quad \frac{d+1}{2} \leq j \leq d\,. \end{cases}$$

Therefore, the first polynomial in a pair is unimodal with peak at $\frac{d+1}{2} - 1$ and the second polynomial in a pair is unimodal with peak at $\frac{d+1}{2}$. After adding the two polynomials in a pair, the coefficient of $t^{\frac{d+1}{2}-1}$ will be equal to the coefficient of $t^{\frac{d+1}{2}}$ and the resulting sum will be unimodal with double peak at $\frac{d+1}{2} - 1$ and $\frac{d+1}{2}$.

The coefficient $\binom{d}{j}$ is nonnegative for all d and j and thus will not affect the symmetry or the unimodality of the sum of each polynomial pair. It follows that the entire sum in (4.2) is symmetric and unimodal with centers of symmetry in agreement with the peaks of unimodality. Because the peaks are in the middle, the entire sum is alternatingly increasing when d is odd.

Case 2: d is even.

Once again we rearrange $B(d,t)$:

$$\sum_{j=0}^{d} \binom{d}{j} A_{j+1}(d+1,t) = \sum_{j=0}^{\frac{d}{2}-1} \binom{d}{j} \left[A_{j+1}(d+1,t) + A_{d+1-j}(d+1,t)\right] + \binom{d}{\frac{d}{2}} A_{\frac{d}{2}+1}(d+1,t)\,.$$

As before, the sum of each polynomial pair is symmetric with center of symmetry at $\frac{d}{2}$. By Theorem 3.3 we know that $A_{j+1}(d+1,t)$ is unimodal with peak at $\lfloor \frac{d+1}{2} \rfloor = \frac{d}{2}$ for all values of j with $d \geq 2$ and even. By the same reasoning, the polynomial $A_{\frac{d}{2}+1}(d+1,t)$ is also symmetric and unimodal with center of symmetry and peak in agreement at $\frac{d}{2}$. The nonnegative coefficient $\binom{d}{j}$ will not affect the symmetry or the unimodality of $A_{\frac{d}{2}+1}(d+1,t)$ or of the sum of the polynomial pairs. Therefore, the entire sum is symmetric and unimodal with center of symmetry equal to the peak of unimodality. Because the peak is in the middle we conclude that the entire sum is alternatingly increasing. $\square$

Remark. Corollary 4.2 can also be deduced from Proposition 2.17 in [10] and equation (4.1). The proposition states that the δ-vector for a lattice parallelepiped with at least one interior point is alternatingly increasing. Our U_I-induced decomposition of the ±1-cube provides a geometric proof of the result for this specific class of parallelepipeds.

4.2 Half-Open ±1-Cubes

We begin this section by introducing the (B,ℓ)-Eulerian numbers, a refinement of the type-B Eulerian numbers defined by

$$b_\ell(d,k) := \{(\pi,\epsilon) \in B_d : \epsilon_d\pi_d = d+1-\ell \text{ and } \operatorname{natdes}(\pi,\epsilon) = k\} \ ,$$

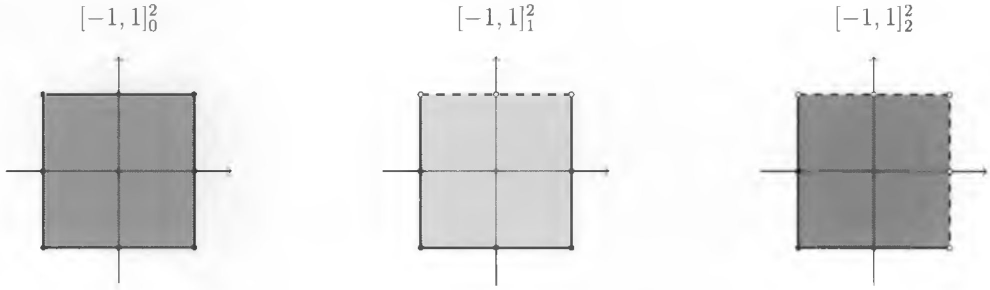


Figure 4.3: Half-open ± 1-cubes $[-1,1]_\ell^2$ for $\ell = 0, 1, 2$, respectively.

where $1 \leq \ell \leq d$. We further introduce the (B,ℓ)**-Eulerian polynomial**,

$$B_\ell(d,t) = \sum_{k=0}^{d} b_\ell(d,k)t^k \,.$$

What else do the (B,ℓ)-Eulerian numbers count? To answer this question we introduce the **half-open ± 1-cube** $[-1,1]_\ell^d$, where $d \geq 1$ and $0 \leq \ell \leq d$. This object is the ± 1-cube with ℓ non-translate facets removed:

$$\begin{aligned}
[-1,1]_\ell^d :&= [-1,1]^d \setminus \{x_d = 1, x_{d-1} = 1, \ldots, x_{d+1-\ell} = 1\} \\
&= \left\{\mathbf{x} \in \mathbb{R}^d : -1 \leq x_i \leq 1 \text{ with } x_i < 1 \text{ when } d+1-\ell \leq i \leq d\right\} \,.
\end{aligned}$$

See Figure 4.3. Additionally we define the **(natural) ℓ-descent set**, $\operatorname{NatDes}_\ell(\pi,\epsilon) \subseteq$

$\{0, 1, \ldots, d\}$, of signed permutation $(\pi, \epsilon) \in B_d$ by

$$\mathrm{NatDes}_\ell(\pi, \epsilon) := \begin{cases} \mathrm{NatDes}(\pi, \epsilon) \cup \{d\} & \text{if } d + 1 - \ell \leq \epsilon_d \pi_d \leq d\,, \\ \mathrm{NatDes}(\pi, \epsilon) & \text{otherwise}\,. \end{cases}$$

The cardinality of this set is the **(natural) ℓ-descent** number of (π, ϵ), denoted by

$$\mathrm{natdes}_\ell(\pi, \epsilon) := |\mathrm{NatDes}_\ell(\pi, \epsilon)|\ .$$

We observe that

$$\mathrm{natdes}_\ell(\pi, \epsilon) = \begin{cases} \mathrm{natdes}(\pi, \epsilon) + 1 & \text{if } d + 1 - \ell \leq \epsilon_d \pi_d \leq d\,, \\ \mathrm{natdes}(\pi, \epsilon) & \text{otherwise}\,. \end{cases}$$

As with $[-1, 1]^d$, we can decompose $[-1, 1]^d_\ell$ into half-open unimodular simplices indexed by signed permutations in B_d. However, we must first adjust our definition of the simplices to account for the changes in the ± 1-cube. Let

$$\triangle^{d,\ell}_{(\pi,\epsilon)} := \left\{ \begin{array}{c} \mathbf{x} \in [-1, 1]^d_\ell : 0 \leq \epsilon_1 x_{\pi_1} \leq \cdots \leq \epsilon_d x_{\pi_d} \leq 1 \\ \text{with } \epsilon_i x_{\pi_i} < \epsilon_{i+1} x_{\pi_{i+1}} \text{ when } i \in \mathrm{NatDes}(\pi, \epsilon) \end{array} \right\}.$$

These simplices are disjoint, and their union is $[-1,1]^d_\ell$:

$$[-1,1]^d_\ell = \bigsqcup_{(\pi,\epsilon)\in B_d} \triangle^{d,\ell}_{(\pi,\epsilon)}.$$

Unlike the half-open simplices $\triangle^d_{(\pi,\epsilon)}$ we introduced earlier, the number of missing facets of $\triangle^{d,\ell}_{(\pi,\epsilon)}$ is not always enumerated by the number of descents of the indexing signed permutation. The number of missing facets of a simplex $\triangle^{d,\ell}_{(\pi,\epsilon)}$ that lies on a removed facet of $[-1,1]^d_\ell$ is one more than its descent number. This follows by construction: $\triangle^{d,\ell}_{(\pi,\epsilon)}$ lies on the hyperplane $\{x_i = 1\}$ if and only if the last signed letter of the permutation is $+i$.

The removed facets of $[-1,1]^d_\ell$ are defined by supporting hyperplanes $\{x_{d+1-\ell} = 1\}, \ldots, \{x_d = 1\}$. Therefore, the simplex $\triangle^{d,\ell}_{(\pi,\epsilon)}$ has $\mathrm{natdes}(\pi,\epsilon)+1$ open facets if and only if $\epsilon_d\pi_d \in \{d+1-\ell,\ldots,d\}$. It follows that the number of open facets of $\triangle^{d,\ell}_{(\pi,\epsilon)}$ is $\mathrm{natdes}_\ell(\pi,\epsilon)$. The ℓ-descent set of (π,ϵ) also determines the location of the strict inequalities in the construction of $\triangle^{d,\ell}_{(\pi,\epsilon)}$ and thus the location of the open facets in the simplex itself. We see this in the following equivalent definition:

$$\triangle^{d,\ell}_{(\pi,\epsilon)} = \left\{ \begin{array}{c} \mathbf{x}\in\mathbb{R}^d : 0 \le \epsilon_1 x_{\pi_1} \le \cdots \le \epsilon_d x_{\pi_d} \le 1 \\ \text{with } \epsilon_i x_{\pi_i} < \epsilon_{i+1} x_{\pi_{i+1}} \text{ when } i \in \mathrm{NatDes}_\ell(\pi,\epsilon) \\ \text{and } \epsilon_d x_{\pi_d} < 1 \text{ when } d \in \mathrm{NatDes}_\ell(\pi,\epsilon) \end{array} \right\}.$$

Consider the following map:

$$\begin{aligned}
\varphi : B_d &\to \{(\pi,\epsilon) \in B_{d+1} : \epsilon_{d+1}\pi_{d+1} = d+1-\ell\} \\
\pi &\mapsto \left(i \mapsto \begin{cases} \pi_i & \text{if } 0 \le \pi_i \le d-\ell\,, \\ \pi_i + 1 & \text{if } d+1-\ell \le \pi_i \le d\,, \\ d+1-\ell & \text{if } i = d+1\,, \end{cases} \right) \\
\epsilon &\mapsto (\epsilon_1, \ldots, \epsilon_d, 1)\,.
\end{aligned}$$

The map φ defines a bijection between signed permutations of order d and signed permutations of order $d+1$ with last signed letter $\epsilon_{d+1}\pi_{d+1} = d+1-\ell$. Furthermore, the descent number of the image is equal to the ℓ-descent number of the pre-image. That is,

$$\operatorname{natdes}(\varphi(\pi,\epsilon)) = \operatorname{natdes}_\ell(\pi,\epsilon)\,. \tag{4.3}$$

In fact,

$$\operatorname{NatDes}_\ell(\pi,\epsilon) = \operatorname{NatDes}(\varphi(\pi,\epsilon))\,. \tag{4.4}$$

We prove these equalities in the following argument.

Let $(\pi,\epsilon) \in B_d$ and $i \in \{0, 1, \ldots, d-1\}$.

Case 1. Suppose the signs of the two adjacent signed letters $\epsilon_i\pi_i$ and $\epsilon_{i+1}\pi_{i+1}$ are different. Then the inequality relation $\prec$ between $\epsilon_i\pi_i$ and $\epsilon_{i+1}\pi_{i+1}$ is determined by the signs of ϵ_i and ϵ_{i+1}. These signs are preserved under the map φ. So

$$\epsilon_i \pi_i \prec \epsilon_{i+1}\pi_{i+1} \iff \varphi(\epsilon_i\pi_i) \prec \varphi(\epsilon_{i+1}\pi_{i+1}) .$$

Case 2. Suppose the signs of the two adjacent signed letters $\epsilon_i\pi_i$ and $\epsilon_{i+1}\pi_{i+1}$ are the same. Then the inequality relation between $\epsilon_i\pi_i$ and $\epsilon_{i+1}\pi_{i+1}$ is preserved if and only if the inequality relation between π_i and π_{i+1} is. In this case we let $\prec$ represent the inequality relation between π_i and π_{i+1}.

- Let $\pi_i, \pi_{i+1} < d+1-\ell$. Then $\varphi(\pi_i) = \pi_i$ and $\varphi(\pi_{i+1}) = \pi_{i+1}$, so

$$\pi_i \prec \pi_{i+1} \iff \varphi(\pi_i) \prec \varphi(\pi_{i+1}) .$$

- Let $\pi_i, \pi_{i+1} \geq d+1-\ell$. Then

$$\pi_i \prec \pi_{i+1} \iff \varphi(\pi_i) = \pi_i + 1 \prec \pi_{i+1} + 1 = \varphi(\pi_{i+1}) .$$

- Let $\pi_i < d+1-\ell$ and $\pi_{i+1} \geq d+1-\ell$. Then $\pi_i < \pi_{i+1}$ and

$$\varphi(\pi_i) = \pi_i < \pi_{i+1} < \pi_{i+1} + 1 = \varphi(\pi_{i+1}) .$$

- Let $\pi_i \geq d+1-\ell$ and $\pi_{i+1} < d+1-\ell$. Then $\pi_i > \pi_{i+1}$ and

$$\varphi(\pi_i) = \pi_i + 1 > \pi_i > \pi_{i+1} = \varphi(\pi_{i+1})\,.$$

We see that the inequality relation $\prec$ between π_i and π_{i+1} is preserved in all instances and so is the inequality relation between $\epsilon_i\pi_i$ and $\epsilon_{i+1}\pi_{i+1}$. From Cases 1 and 2 we conclude that for all $i \in \{0, 1, \ldots, d-1\}$

$$i \in \mathrm{NatDes}(\pi, \epsilon) \qquad \Longleftrightarrow \qquad i \in \mathrm{NatDes}(\varphi(\pi, \epsilon))\,.$$

Now consider the d^{th} signed letter $\epsilon_d\pi_d$ of $(\pi, \epsilon) \in B_d$. Suppose $\epsilon_d = 1$ and $d+1-\ell \leq \pi_d \leq d$. This implies $d \in \mathrm{NatDes}_\ell(\pi, \epsilon)$. The d^{th} signed letter of $\varphi(\pi, \epsilon)$ is

$$\varphi(\epsilon_d\pi_d) = \epsilon_d(\pi_d + 1) = \pi_d + 1\,.$$

The $(d+1)^{\text{st}}$ signed letter of $\varphi(\pi, \epsilon)$ is $d+1-\ell$ by definition. We assumed $d+1-\ell \leq \pi_d \leq d$, so $d+1-\ell < \pi_d + 1$. This implies $d \in \mathrm{NatDes}(\varphi(\pi, \epsilon))$.

Now suppose $\epsilon_d = 1$ and $d+1-\ell \leq \pi_d \leq d$ do not both hold. This implies $d \notin \mathrm{NatDes}_\ell(\pi, \epsilon)$. We consider two cases.

a) Suppose $\epsilon_d = -1$. Then

$$\varphi(\epsilon_d\pi_d) = -\varphi(\pi_d) < 0 < d+1-\ell\,.$$

From this we see that $d \notin \mathrm{NatDes}(\varphi(\pi,\epsilon))$.

b) Suppose $\epsilon_d = 1$ and $\pi_d < d+1-\ell$. It follows that $\varphi(\epsilon_d\pi_d) = \pi_d$. The $(d+1)^{\text{st}}$ signed letter of $\varphi(\pi,\epsilon)$ is $d+1-\ell$, so $d \notin \mathrm{NatDes}(\varphi(\pi,\epsilon))$.

All implications are bi-directional. Therefore,

$$d \in \mathrm{NatDes}_\ell(\pi,\epsilon) \qquad \Longleftrightarrow \qquad d \in \mathrm{NatDes}(\varphi(\pi,\epsilon))\,.$$

Together with Cases 1 and 2 we now conclude that i is an ℓ-descent of (π,ϵ) if and only if i is a descent of $\varphi(\pi,\epsilon)$. Therefore,

$$\begin{aligned}\mathrm{NatDes}_\ell(\pi,\epsilon) &= \mathrm{NatDes}\left(\varphi(\pi,\epsilon)\right) \text{ and}\\ \mathrm{natdes}_\ell(\pi,\epsilon) &= \mathrm{natdes}\left(\varphi(\pi,\epsilon)\right)\,.\end{aligned}$$

This gives us the following lemma.

Lemma 4.3. *Let $d \geq 1$ and $0 \leq \ell \leq d$. Then*

$$b_{\ell+1}(d+1,k) = \left|\{(\pi,\epsilon) \in B_d : \mathrm{natdes}_\ell(\pi,\epsilon) = k\}\right|\,.$$

Theorem 4.4.

$$\mathsf{Ehr}\left([-1,1]^d_\ell, t\right) = \frac{B_{\ell+1}(d+1,t)}{(1-t)^{d+1}}\,.$$

Proof. We saw above that the number of open facets of $\triangle^{d,\ell}_{(\pi,\epsilon)}$ is the ℓ-descent number of (π, ϵ). We also have the disjoint decomposition

$$[-1,1]^d_\ell = \bigsqcup_{(\pi,\epsilon)\in B_d} \triangle^{d,\ell}_{(\pi,\epsilon)} .$$

Therefore,

$$\begin{aligned}
\mathsf{Ehr}\left([-1,1]^d_\ell, t\right) &= \sum_{(\pi,\epsilon)\in B_d} \mathsf{Ehr}\left(\triangle^{d,\ell}_{(\pi,\epsilon)}, t\right) \\
&= \sum_{(\pi,\epsilon)\in B_d} \frac{\delta\left(\triangle^{d,\ell}_{(\pi,\epsilon)}, t\right)}{(1-t)^{d+1}} \\
&= \frac{\sum_{(\pi,\epsilon)\in B_d} t^{\mathrm{natdes}_\ell(\pi,\epsilon)}}{(1-t)^{d+1}} \\
&= \frac{\sum_{k=0}^{d} b_{\ell+1}(d+1,k)t^k}{(1-t)^{d+1}} \\
&= \frac{B_{\ell+1}(d+1,t)}{(1-t)^{d+1}} .
\end{aligned}$$

□

Theorem 4.5. *For $d \geq 1$ and $0 \leq \ell \leq d$, the coefficients of $\delta\left([-1,1]^d_\ell, t\right)$ are alternatingly increasing.*

In order to prove Theorem 4.5, we first introduce another disjoint decomposition of $[-1,1]^d_\ell$. From the resulting geometry we obtain a characterization for $\delta\left([-1,1]^d_\ell, t\right)$ as a linear combination of (A,j)-Eulerian polynomials. We then use the symmetric, unimodal and recursive properties of the nonnegative integral coefficients of the linear combination to prove that the coefficients of $\delta\left([-1,1]^d_\ell, t\right)$ are alternatingly increasing.

Consider the U_I-induced decomposition of $[-1,1]^d_\ell$ into **half-open unit cells**:

$$[-1,1]^d_\ell = \bigsqcup_{I\subseteq[d]} U_I^{d,\ell},$$

where

$$\begin{aligned} U_I^{d,\ell} &:= \left\{\mathbf{x}\in[-1,1]^d_\ell : x_i \geq 0 \text{ if } i\in I \text{ and } x_i<0 \text{ if } i\notin I\right\} \\ &= \left\{\begin{array}{c} \mathbf{x}\in\mathbb{R}^d : 0\leq x_i\leq 1 \text{ for all } i\in I \text{ with} \\ x_i<1 \text{ when } d+1-l\leq i\leq d, \\ \text{and } -1\leq x_i<0 \text{ for all } i\notin I \end{array}\right\}. \end{aligned}$$

By construction, the decomposition is disjoint so the δ-polynomial of the half-open ± 1-cube is equal to the sum of the δ-polynomials of the half-open unit cells. Furthermore, $U_I^{d,\ell}$ is congruent to C_j^d, where

$$j = |[d]\setminus I| + |\{i\in I : d+1-\ell\leq i\leq d\}|\ .$$

Let $I\subseteq[d]$, then the unit cell $U_I^{d,\ell}$ has $|[d]\setminus I| = d-|I|$ missing facets before the ℓ facets of the ± 1-cube are removed. This corresponds to the strict inequalities at 0 in the definition of $U_I^{d,\ell}$. After the ℓ facets of the ± 1-cube are removed, $\{x_d = 1\},\ldots,\{x_{d+1-\ell}=1\}$, we see strict inequalities at 1 for all x_i with $i\in I$ and $d+$

$1-\ell \leq i \leq d$. These correspond to one missing facet each, giving us a total of $j = |[d] \setminus I| + |\{i \in I : d+1-\ell \leq i \leq d\}|$ missing facets.

It follows that the δ-polynomial of $U_I^{d,\ell}$ *is* the δ-polynomial of C_j^d and

$$\delta\left([-1,1]_\ell^d, t\right) = \sum_{I \subseteq [d]} \delta\left(U_I^{d,\ell}, t\right) = \sum_{j=0}^{d} c_j^{d,\ell} \cdot \delta\left(C_j^d, t\right) = \sum_{j=0}^{d} c_j^{d,\ell} \cdot A_{j+1}(d+1,t),$$

where

$$c_j^{d,\ell} := \left|\left\{I \subseteq [d] : U_I^{d,\ell} \cong C_j^d\right\}\right| .$$

By Lemma 3.4 we know that the coefficients of $A_{j+1}(d+1,t)$ are alternatingly increasing for all $j \geq \left\lfloor \frac{d+1}{2} \right\rfloor$. Additionally, from the proof of Corollary 4.2 we know that for all $0 \leq j \leq d$, the sum of the polynomial pair $A_{j+1}(d+1,t)$, $A_{d-j+1}(d+1,t)$ is alternatingly increasing.

In order to prove that $\delta\left([-1,1]_\ell^d, t\right)$ is alternatingly increasing we will show that for each $I \subseteq [d]$ with $U_I^{d,\ell} \cong C_j^d$ and $j < \left\lfloor \frac{d+1}{2} \right\rfloor$, there exists $I' \subseteq [d]$ with $U_{I'}^{d,\ell} \cong C_{d-j}^d$. This will allow us to pair each (A,j)-polynomial where $j < \left\lfloor \frac{d+1}{2} \right\rfloor$ with a second polynomial such that the polynomial sum is alternatingly increasing. The index sets not paired in this process correspond to j-values greater than or equal to $\left\lfloor \frac{d+1}{2} \right\rfloor$ and thus the associated (A,j)-polynomials are alternatingly increasing on their own. In this way, we will see that the sum of the (A,j)-polynomials, $\sum_{j=0}^{d} c_j^{d,\ell} \cdot A_{j+1}(d+1,t)$, is itself an alternatingly increasing polynomial.

It is important to note that we consider all (A,j)-polynomials $A_{j+1}(d+1,t)$ as degree-d polynomials, although the coefficient of t^d may in fact be zero. Furthermore, we use the term "alternatingly increasing" in its strictest sense. That is, when we say that a degree-d polynomial is alternatingly increasing, we mean that the polynomial is alternatingly increasing in degree d (whether or not $s < d$, where s is the largest index with a non-zero coefficient). Symbolically, if $a_0 + a_1 t + \cdots + a_s t^s + \cdots + a_d t^d$ is alternatingly increasing, then

$$a_0 \leq a_d \leq a_1 \leq \cdots \leq a_s \leq \cdots \leq a_{\lfloor \frac{d+1}{2} \rfloor}.$$

For fixed $d \geq 1$ and fixed ℓ such that $0 \leq \ell \leq d$, let

$$\left(c_0^{d,\ell}, c_1^{d,\ell}, \ldots, c_d^{d,\ell}\right)$$

be called the $C_j^{d,\ell}$**-vector**. From the definition of $c_k^{d,\ell}$ above, we see that the k^{th} entry of this vector is equal to the number of unit cells congruent to C_k^d in the U_I-induced decomposition of $[-1,1]_\ell^d$. For $j < 0$ and $j > d$, we define $c_j^{d,\ell} := 0$. This aligns with the combinatorial and geometric interpretations of $c_j^{d,\ell}$. To get a sense of what these $C_j^{d,\ell}$-vectors look like, see Table 4.1.

In our proof of Theorem 4.5 we will need the following lemma regarding the $C_j^{d,\ell}$-vector.

Lemma 4.6. *Let $d \geq 1$, $j \in \mathbb{Z}$ and $0 \leq \ell \leq d$. Then the $C_j^{d,\ell}$-vector is symmetric*

		$c_0^{d,\ell}$	$c_1^{d,\ell}$	$c_2^{d,\ell}$	$c_3^{d,\ell}$	$c_4^{d,\ell}$
$d=1$	$\ell=0$	1	1			
$d=1$	$\ell=1$	0	2			
$d=2$	$\ell=0$	1	2	1		
$d=2$	$\ell=1$	0	2	2		
$d=2$	$\ell=2$	0	0	4		
$d=3$	$\ell=0$	1	3	3	1	
$d=3$	$\ell=1$	0	2	4	2	
$d=3$	$\ell=2$	0	0	4	4	
$d=3$	$\ell=3$	0	0	0	0	8
$d=4$	$\ell=0$	1	4	6	4	1
$d=4$	$\ell=1$	0	2	6	6	2
$d=4$	$\ell=2$	0	0	4	8	4
$d=4$	$\ell=3$	0	0	0	8	8
$d=4$	$\ell=4$	0	0	0	0	16

Table 4.1: The $C_j^{d,\ell}$-vector for $d = 1, 2, 3, 4$.

and unimodal with center of symmetry at $\lfloor \frac{d+\ell}{2} \rfloor$ and peak(s) of unimodality at:

$$\begin{cases} \frac{d+\ell}{2} & \textit{when } d+\ell \textit{ is even}, \\ \lfloor \frac{d+\ell}{2} \rfloor \textit{ and } \lfloor \frac{d+\ell}{2} \rfloor + 1 & \textit{when } d+\ell \textit{ is odd}. \end{cases}$$

In particular, the center of symmetry is in agreement with the peak(s) of unimodality.

Proof. First we will prove that the $C_j^{d,\ell}$-vector is symmetric about $\lfloor \frac{d+\ell}{2} \rfloor$ by showing a bijection between unit cells in $[-1,1]^d_\ell$ congruent to C_j^d and unit cells in $[-1,1]^d_\ell$ congruent to $C^d_{d+\ell-j}$. We will then use induction to prove that the $C_j^{d,\ell}$-vector is unimodal with peak(s) of unimodality in agreement with the center of symmetry.

Fix $d \geq 1$ and ℓ such that $0 \leq \ell \leq d$. In order to prove symmetry of the $C_j^{d,\ell}$-

vector we will show that $U_I^{d,\ell} \cong C_j^d$ if and only if $U_{\hat{I}}^{d,\ell} \cong C_{d+\ell-j}^d$, where $\hat{I} = [d] \setminus I$.

Suppose the index set $I \subseteq [d]$ is such that $U_I^{d,\ell} \cong C_j^d$. We know $j = |[d] \setminus I| + |\{i \in I : d+1-\ell \leq i \leq d\}|$. Let $F := \{i \in I : d+1-\ell \leq i \leq d\}$. Then

$$j = d - |I| + |F| \,.$$

Now consider the index set $\hat{I} = [d] \setminus I$. Notice that $|\hat{I}| = d - |I|$. Thus the unit cell $U_{\hat{I}}^{d,\ell}$ has

$$\hat{j} = d - |\hat{I}| + |\hat{F}| = d - (d - |I|) + |\hat{F}| = |I| + |\hat{F}|$$

removed facets.

We note that F and $\hat{F}$ are disjoint because I and $\hat{I}$ are disjoint. We further note that the union of F and $\hat{F}$ is the set of indices corresponding to all removed facets of $[-1,1]_\ell^d$. So

$$F \sqcup \hat{F} = \{d+1-\ell, \ldots, d-1, d\} \quad \Longrightarrow \quad |F| + |\hat{F}| = \ell \,.$$

Therefore,

$$\begin{aligned} j &= d - |I| + |F| \\ -j &= -d + |I| - |F| \\ d + \ell - j &= |I| + (\ell - |F|) \\ d + \ell - j &= |I| + |\hat{F}|\,, \end{aligned}$$

from which it follows

$$\hat{j} = d + \ell - j \quad \text{and} \quad U^{d,\ell}_{\hat{I}} \cong C^d_{d+\ell-j}\,.$$

We conclude $U^{d,\ell}_I \cong C^d_j$ if and only if $U^{d,\ell}_{I\setminus[d]} \cong C^d_{d+\ell-j}$. This means $c^{d,\ell}_j = c^{d,\ell}_{d+\ell-j}$ for all $j \in \mathbb{Z}$. It further implies the $C^{d,\ell}_j$-vector is symmetric about

$$\frac{j + (d + \ell - j)}{2} = \frac{d + \ell}{2}\,.$$

We will prove the unimodality of the $C^{d,\ell}_j$-vector for all $0 \le \ell \le d$ by induction on d. When $d = 1$, we have $C^{1,\ell}_j$-vectors $(1, 1)$ and $(0, 2)$ for $\ell = 0$ and $\ell = 1$, respectively. In this case the peaks of unimodality are trivial. When $d = 2$, we have $C^{2,\ell}_j$-vectors $(1, 2, 1)$, $(0, 2, 2)$ and $(0, 0, 4)$ for $\ell = 0, 1, 2$, respectively. See Figure 4.4. We observe a single peak of unimodality at index $\frac{2+\ell}{2}$ when $2 + \ell$ is even, and a

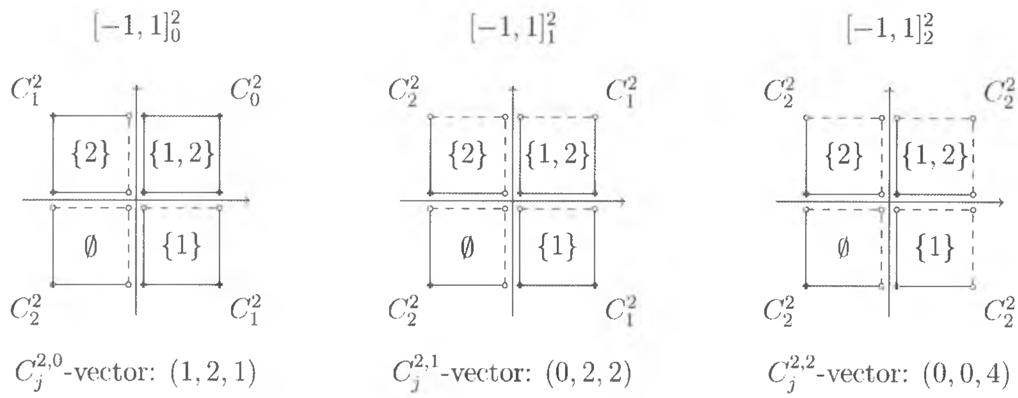


Figure 4.4: Decomposition of $[-1,1]^2_\ell$ into unit cells indexed by subsets of $\{1,2\}$ for $\ell = 0, 1, 2$.

double peak of unimodality at indices $\lfloor \frac{2+\ell}{2} \rfloor$ and $\lfloor \frac{2+\ell}{2} \rfloor + 1$ when $2+\ell$ is odd.

Let $d \geq 2$. Suppose the $C^{d-1,\ell}_j$-vectors are unimodal for all ℓ satisfying $0 \leq \ell \leq d-1$ with the desired peak(s) of unimodality. We will proceed by cases on ℓ.

Case 1: $\ell = 0$ **.**

Let $I' \subseteq [d-1]$. In dimension d, $U^{d,0}_{I'} \cong C^d_j$ where $j = |[d] \setminus I'| = d - |I'|$. In dimension $d-1$, $U^{d-1,0}_{I'} \cong C^{d-1}_{j-1}$ because $|[d-1] \setminus I'| = d-1-|I'| = j-1$. Therefore, when $K \subseteq [d]$ such that $d \notin K$,

$$U^{d,0}_K \cong C^d_j \quad \iff \quad U^{d-1,0}_K \cong C^{d-1}_{j-1}\,.$$

Let $I = I' \cup \{d\}$. In dimension d, $U^{d,0}_I \cong C^d_j$ where $j = |[d] \setminus I| = d - |I|$. In dimension $d-1$, the index $i = d$ does not contribute to the geometry of the unit cell so $U_I^{d-1,0} \cong U_{I'}^{d-1,0} \cong C_j^{d-1}$ because

$$|[d-1] \setminus I| = d - 1 - (|I| - 1) = d - |I| = j\,.$$

Therefore, when $K \subseteq [d]$ such that $d \in K$,

$$U_K^{d,0} \cong C_j^d \qquad \Longleftrightarrow \qquad U_K^{d-1,0} \cong C_j^{d-1}\,.$$

It follows that

$$\begin{aligned}
c_j^{d,0} &= \left|\left\{ I \subseteq [d] : U_I^{d,0} \cong C_j^d \right\}\right| \\
&= \left|\left\{ I \subseteq [d] : d \notin I \text{ and } U_I^{d,0} \cong C_j^d \right\}\right| + \left|\left\{ I \subseteq [d] : d \in I \text{ and } U_I^{d,0} \cong C_j^d \right\}\right| \\
&= \left|\left\{ I' \subseteq [d-1] : U_{I'}^{d-1,0} \cong C_{j-1}^{d-1} \right\}\right| + \left|\left\{ I' \subseteq [d-1] : U_{I'}^{d-1,0} \cong C_j^{d-1} \right\}\right| \\
&= c_{j-1}^{d-1,0} + c_j^{d-1,0}\,.
\end{aligned}$$

By symmetry, to prove the unimodality of the $C_j^{d,0}$-vector, it is sufficient to show

$$c_j^{d,0} \leq c_{j+1}^{d,0} \tag{4.5}$$

holds for all $j < \lfloor \frac{d}{2} \rfloor$.

Let $j < \left\lfloor \frac{d}{2} \right\rfloor$. We note that inequality (4.5) is equivalent to

$$c_{j-1}^{d-1,0} + c_j^{d-1,0} \leq c_j^{d-1,0} + c_{j+1}^{d-1,0}$$

and thus to

$$c_{j-1}^{d-1,0} \leq c_{j+1}^{d-1,0}\,. \tag{4.6}$$

By the induction hypothesis, if d is odd, then the (single) peak of unimodality of the $C_j^{d-1,0}$-vector is $\frac{d-1}{2}$ which implies $c_k^{d-1,0} \leq c_{k+1}^{d-1,0}$ for all $k < \frac{d-1}{2}$. We note $j < \left\lfloor \frac{d}{2} \right\rfloor$ is equivalent to $j+1 \leq \frac{d-1}{2}$ when d is odd. Therefore, (4.6) holds by the induction hypothesis and (4.5) holds for all $j < \left\lfloor \frac{d}{2} \right\rfloor$ when d is odd.

If d is even, then the (double) peaks of unimodality of the $C_j^{d-1,0}$-vector are $\left\lfloor \frac{d-1}{2} \right\rfloor$ and $\left\lfloor \frac{d-1}{2} \right\rfloor + 1 = \left\lfloor \frac{d+1}{2} \right\rfloor = \frac{d}{2}$. It follows that $c_k^{d-1,0} \leq c_{k+1}^{d-1,0}$ for all $k < \frac{d}{2}$, or equivalently, for all $k+1 \leq \frac{d}{2}$. We still have $j < \left\lfloor \frac{d}{2} \right\rfloor$ which is equivalent to $j+1 \leq \frac{d}{2}$ when d is even. Therefore, (4.6) holds by the induction hypothesis, implying that (4.5) holds for all $j < \left\lfloor \frac{d}{2} \right\rfloor$ when d is even.

By the symmetry of the $C_j^{d,0}$-vector we have

$$c_j^{d,0} \geq c_{j+1}^{d,0}$$

for all $j \geq \left\lfloor \frac{d}{2} \right\rfloor$. We conclude that the $C_j^{d,0}$-vector is unimodal for $d \geq 1$ and $0 \leq j \leq d$.

Remark. From Theorem 4.1 we know $c_j^{d,0} = \binom{d}{j}$ for all $d \geq 1$ and $j \in \mathbb{Z}$. This gives us the $C_j^{d,0}$-vector

$$\left(\binom{d}{0}, \binom{d}{1}, \ldots, \binom{d}{d}\right).$$

Common knowledge about the binomial coefficients tells us that this vector is palindromic and unimodal and thus alternatingly increasing. Case 1 gives a geometric interpretation of this result in addition to a geometric interpretation of the well-known recursive formula for binomial coefficients:

$$c_j^{d,0} = c_{j-1}^{d-1,0} + c_j^{d-1,0} \quad \Longleftrightarrow \quad \binom{d}{j} = \binom{d-1}{j} + \binom{d-1}{j-1}.$$

Case 2: $\ell > 0$ **.**

Let $I' \subseteq [d-1]$. In dimension d, $U_{I'}^{d,\ell} \cong C_j^d$ where

$$j = |[d] \setminus I'| + |\{i \in I' : d + 1 - \ell \leq i \leq d\}| \, .$$

By definition, this unit cell is a subset of $[-1, 1]_\ell^d$, a half-open ± 1-cube with ℓ facets missing corresponding to the ℓ supporting hyperplanes $x_d = 1, x_{d-1} = 1, \ldots, x_{d+1-\ell} = 1$.

If we consider this same ± 1-cube in one dimension lower, then we observe $\ell - 1$ missing facets corresponding to supporting hyperplanes $x_{d-1} = 1, \ldots, x_{d+1-\ell} = 1$. Therefore, when considered as a $(d-1)$-dimensional object, the unit cell indexed by

I' is a subset of $[-1,1]^{d-1}_{\ell-1}$ and is congruent to C^{d-1}_k where

$$\begin{aligned} k &= |[d-1] \setminus I'| + |\{i \in I' : (d-1)+1-(\ell-1) \le i \le d-1\}| \\ &= (d-1) - |I'| + |\{i \in I' : d+1-\ell \le i \le d-1\}| \\ &= |[d] \setminus I'| - 1 + |\{i \in I' : d+1-\ell \le i \le d\}| \\ &= j-1\,. \end{aligned}$$

Therefore, when $K \subseteq [d]$ such that $d \notin K$,

$$U^{d,\ell}_K \cong C^d_j \qquad \Longleftrightarrow \qquad U^{d-1,\ell-1}_K \cong C^{d-1}_{j-1}\,.$$

Let $I = I' \cup \{d\}$. In dimension d, $U^{d,\ell}_I \cong C^d_j$ where

$$j = |[d] \setminus I| + |\{i \in I : d+1-\ell \le i \le d\}|\,.$$

In dimension $d-1$, we consider the unit cell indexed by I as a subset of $[-1,1]^{d-1}_{\ell-1}$ just as before. Therefore, $U^{d,\ell}_I \cong U^{d,\ell-1}_{I'} \cong C^{d-1}_{j-1}$ in dimension $d-1$ because

$$\begin{aligned} j-1 &= d - |I| + |\{i \in I : d+1-\ell \le i \le d\}| - 1 \\ &= |[d-1] \setminus I'| + |\{i \in I' : d+1-\ell \le i \le d-1\}|\,. \end{aligned}$$

Therefore, when $K \subseteq [d]$ such that $d \in K$,

$$U_K^{d,\ell} \cong C_j^d \quad \iff \quad U_K^{d-1,\ell-1} \cong C_{j-1}^{d-1}.$$

It follows that

$$\begin{aligned}
c_j^{d,\ell} &= \left|\left\{I \subseteq [d] : U_I^{d,\ell} \cong C_j^d\right\}\right| \\
&= \left|\left\{I \subseteq [d] : d \notin I \text{ and } U_I^{d,\ell} \cong C_j^d\right\}\right| + \left|\left\{I \subseteq [d] : d \in I \text{ and } U_I^{d,\ell} \cong C_j^d\right\}\right| \\
&= \left|\left\{I' \subseteq [d-1] : U_{I'}^{d-1,\ell-1} \cong C_{j-1}^{d-1}\right\}\right| + \left|\left\{I' \subseteq [d-1] : U_{I'}^{d-1,\ell-1} \cong C_{j-1}^{d-1}\right\}\right| \\
&= 2 \cdot c_{j-1}^{d-1,\ell-1}.
\end{aligned}$$

Suppose $j < \left\lfloor \frac{d+\ell}{2} \right\rfloor$. We wish to show

$$c_j^{d,\ell} \leq c_{j+1}^{d,\ell}, \tag{4.7}$$

or equivalently,

$$c_{j-1}^{d-1,\ell-1} \leq c_j^{d-1,\ell-1}. \tag{4.8}$$

By the induction hypothesis, inequality (4.8) holds for all $\ell > 0$ when $j - 1 < \left\lfloor \frac{d-1+\ell}{2} \right\rfloor$. Equivalently, inequality (4.8) holds for all $\ell > 0$ when $j < \left\lfloor \frac{d+1+\ell}{2} \right\rfloor$. However, $\left\lfloor \frac{d+\ell}{2} \right\rfloor \leq \left\lfloor \frac{d+1+\ell}{2} \right\rfloor$. So $j < \left\lfloor \frac{d+\ell}{2} \right\rfloor$ implies (4.8) which further implies (4.7). From the symmetry of the $C_j^{d,\ell}$-vector,

$$c_j^{d,\ell} \geq c_{j+1}^{d,\ell}$$

for all $j \geq \left\lfloor \frac{d+1+\ell}{2} \right\rfloor$. Therefore, the $C_j^{d,\ell}$-vector is unimodal for $d \geq 1$, $0 \leq j \leq d$ and $\ell > 0$.

This proves the unimodality of the $C_j^{d,\ell}$-vector. □

We are now equipped to prove that the coefficients of $\delta\left([-1,1]_\ell^d, t\right)$ are alternatingly increasing.

Proof of Theorem 4.5. Fix $d \geq 1$ and ℓ such that $0 \leq \ell \leq d$. Suppose $j < \left\lfloor \frac{d+1}{2} \right\rfloor$. We wish to show that

$$c_j^{d,\ell} \leq c_{d-j}^{d,\ell}. \tag{4.9}$$

First note $j < \left\lfloor \frac{d+1}{2} \right\rfloor$ implies $j < d - j$. If j and $d - j$ are both less than or equal to $p_1 := \left\lfloor \frac{d+\ell}{2} \right\rfloor$, then inequality (4.9) holds by unimodality of the $C_j^{d,\ell}$-vector: $c_k^{d,\ell} \leq c_{k',\ell}^{d}$ for all $k \leq k' \leq p_1$.

If $j \leq p_1 < d - j$, then we need to show that the index $d - j$ is closer to $p_1 + 1$ than the index j is to p_1. That is, we need to show

$$p_1 - j \geq (d - j) - (p_1 + 1),$$

or equivalently,

$$2p_1 \geq d - 1\,.$$

Inequality (4.9) will follow from the symmetry of the $C_j^{d,\ell}$-vector.

When $d + \ell$ is even we have

$$2p_1 = 2\left(\frac{d+\ell}{2}\right) = d + \ell > d - 1\,.$$

When $d + \ell$ is odd we have

$$2p_1 = 2\left(\frac{d+\ell}{2} - \frac{1}{2}\right) = d + \ell - 1 \geq d - 1\,.$$

Therefore,

$$2p_1 \geq d - 1$$

holds for all $d \geq 1$ and $0 \leq \ell \leq d$.

We conclude that the number of unit cells congruent to C_j^d in the U_I-induced decomposition of $[-1,1]_\ell^d$ is less than or equal to the number of unit cells congruent to C_{d-j}^d in the decomposition. Therefore we can pair each of the (A, j)-polynomials with j-value strictly less than $\left\lfloor \frac{d+1}{2} \right\rfloor$ with one of the $(A, d-j)$-polynomials such that no $(A, d-j)$-polynomial is paired twice. The sum of each polynomial pair is alternatingly increasing. Furthermore, any (A, j)-polynomial not paired in this process is itself alternatingly increasing. Thus, for fixed $d \geq 1$ and $0 \leq \ell \leq d$, the coefficients of

$$\delta\left([-1,1]^d_\ell, t\right) = \sum_{j=0}^{d} c_j^{d,\ell} \cdot A_{j+1}(d+1,t)$$

are alternatingly increasing. □

We arrive at the following corollary immediately.

Corollary 4.7. *$B_{\ell+1}(d+1,t)$ is alternatingly increasing for $d \geq 1$ and $0 \leq \ell \leq d$.*

Chapter 5

Lattice Centrally Symmetric Parallelepipeds

5.1 Parallelepipeds with Lattice Centrally Symmetric Edges

A polytope $\mathcal{P}$ is **centrally symmetric about the origin** if $\mathbf{p} \in \mathcal{P}$ implies $-\mathbf{p} \in \mathcal{P}$. A polytope $\mathcal{P}$ is **lattice centrally symmetric** if there exists $\mathbf{v} \in \mathbb{Z}^d$ such that $\mathbf{v} + \mathcal{P}$ is centrally symmetric about the origin. Moreover, we say the half-open lattice parallelepiped $\Diamond_I(J)$ is lattice centrally symmetric if its closure $\Diamond(J)$ is.

The ± 1-cube is a simple example of a lattice centrally symmetric parallelepiped. We have already seen that the δ-vector for the ± 1-cube (closed and half-open) is alternatingly increasing.

Theorem 5.1. *Let $\Diamond_I(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ be a half-open parallelepiped with lattice centrally* *symmetric edges. Then*

$$\mathsf{Ehr}\left(\lozenge_I\left(\mathbf{v}_1,\ldots,\mathbf{v}_d\right),t\right)=\frac{\sum_{K\subseteq[d]}\#\left(\square(\mathbf{w}_{i_1},\ldots,\mathbf{w}_{i_{|K|}})\cap\mathbb{Z}^d\right)B_{|I\cup K|+1}(d+1,t)}{(1-t)^{d+1}},$$

where $\mathbf{w}_i=\frac{1}{2}\mathbf{v}_i\in\mathbb{Z}^d$ *for all* $i\in[d]$.

Proof. Let $\lozenge_I(\mathbf{v}_1,\ldots,\mathbf{v}_d)$ be a half-open parallelepiped with lattice centrally symmetric edges. Then there exists $\mathbf{w}_i\in\mathbb{Z}^d$ such that $\mathbf{w}_i=\frac{1}{2}\mathbf{v}_i$. In this way we see that

$$\lozenge_I(\mathbf{v}_1,\ldots,\mathbf{v}_d)=\lozenge_I(2\mathbf{w}_1,\ldots,2\mathbf{w}_d)=2\lozenge_I(\mathbf{w}_1,\ldots,\mathbf{w}_d).$$

We also note that $[-1,1]^d_\ell\cong 2\,C^d_\ell$. This implies

$$\begin{aligned}\mathsf{ehr}\left([-1,1]^d_\ell,n\right)&=\mathsf{ehr}\left(2\,C^d_\ell,n\right)\\&=\mathsf{ehr}\left(C^d_\ell,2n\right)\\&=\sum_{[\ell]\subseteq J\subseteq[d]}(2n)^{|J|},\end{aligned}$$

where the last equality follows from Corollary 3.9. This allows us to express the

Ehrhart series of $\Diamond_I(\mathbf{v}_1, \ldots, \mathbf{v}_d)$ in terms of the Ehrhart series of half-open ± 1-cubes:

$$
\begin{aligned}
&\mathsf{Ehr}\left(\Diamond_I(\mathbf{v}_1, \ldots, \mathbf{v}_d), t\right) \\
&\qquad = \mathsf{Ehr}\left(2\Diamond_I(\mathbf{w}_1, \ldots, \mathbf{w}_d), t\right) \\
&\qquad = 1 + \sum_{n \geq 1} \#\left(2n\Diamond_I(\mathbf{w}_1, \ldots, \mathbf{w}_d) \cap \mathbb{Z}^d\right) t^n \\
&\qquad = \sum_{n \geq 0} t^n \sum_{I \subseteq J \subseteq [d]} (2n)^{|J|} \#\left(\Pi(\mathbf{w}_{i_1}, \ldots, \mathbf{w}_{i_{|J|}}) \cap \mathbb{Z}^d\right) \text{ by Lemma 3.8} \\
&\qquad = \sum_{n \geq 0} t^n \sum_{I \subseteq J \subseteq [d]} (2n)^{|J|} \sum_{K \subseteq J} \#\left(\square(\mathbf{w}_{i_1}, \ldots, \mathbf{w}_{i_{|K|}}) \cap \mathbb{Z}^d\right) \text{ by Lemma 3.7} \\
&\qquad = \sum_{K \subseteq [d]} \#\left(\square(\mathbf{w}_{i_1}, \ldots, \mathbf{w}_{i_{|K|}}) \cap \mathbb{Z}^d\right) \sum_{n \geq 0} t^n \sum_{I \cup K \subseteq J} (2n)^{|J|} \\
&\qquad = \sum_{K \subseteq [d]} \#\left(\square(\mathbf{w}_{i_1}, \ldots, \mathbf{w}_{i_{|K|}}) \cap \mathbb{Z}^d\right) \sum_{n \geq 0} \mathsf{ehr}\left([-1,1]^d_{|I \cup K|}, n\right) t^n \\
&\qquad = \sum_{K \subseteq [d]} \#\left(\square(\mathbf{w}_{i_1}, \ldots, \mathbf{w}_{i_{|K|}}) \cap \mathbb{Z}^d\right) \mathsf{Ehr}\left([-1,1]^d_{|I \cup K|}, t\right) \\
&\qquad = \frac{\sum_{K \subseteq [d]} \#\left(\square(\mathbf{w}_{i_1}, \ldots, \mathbf{w}_{i_{|K|}}) \cap \mathbb{Z}^d\right) B_{|I \cup K|+1}(d+1, t)}{(1-t)^{d+1}},
\end{aligned}
$$

where the last equality follows from Theorem 4.4. $\square$

Corollary 5.2. *The δ-polynomial of a half-open parallelepiped with lattice centrally symmetric edges has alternatingly increasing coefficients.*

Proof. From Theorem 5.1 we have

$$\delta\left(\Diamond_I(\mathbf{v}_1,\ldots,\mathbf{v}_d),t\right)=\sum_{K\subseteq[d]}\#\left(\Box(\mathbf{w}_{i_1},\ldots,\mathbf{w}_{i_{|K|}})\cap\mathbb{Z}^d\right)B_{|I\cup K|+1}(d+1,t)\,,$$

where $\Diamond_I(\mathbf{v}_1,\ldots,\mathbf{v}_d)$ is a half-open parallelepiped with lattice centrally symmetric edges and $\mathbf{w}_i=\frac{1}{2}\mathbf{v}_i\in\mathbb{Z}^d$. Furthermore, $\#\left(\Box(\mathbf{w}_{i_1},\ldots,\mathbf{w}_{i_{|K|}})\cap\mathbb{Z}^d\right)\geq 0$ for all $K\subseteq[d]$. By Corollary 4.7, the coefficients of $B_{|I\cup K|+1}(d+1,t)$ are alternatingly increasing. The same is true for the coefficients of any nonnegative linear combination of the $B_{|I\cup K|+1}(d+1,t)$ where $K\subseteq[d]$. □

Corollary 5.3. *The δ-polynomial of a zonotope with lattice centrally symmetric edges has alternatingly increasing coefficients.*

Proof. Let $\mathcal{Z}(\mathbf{u}_1,\ldots,\mathbf{u}_r)$ be a d-dimensional zonotope with lattice centrally symmetric edges. Then there exists $\mathbf{w}_i\in\mathbb{Z}^d$ such that $\mathbf{w}_i=\frac{1}{2}\mathbf{u}_i$ for all $i\in[r]$ and

$$\mathcal{Z}(\mathbf{u}_1,\ldots,\mathbf{u}_r)=\mathcal{Z}(2\mathbf{w}_1,\ldots,2\mathbf{w}_r)\ .$$

Let $\mathscr{C}$ be a subdivision of $\mathcal{Z}$ into parallelepipeds whose maximal cells $\Diamond_1,\Diamond_2,\ldots,\Diamond_s$ are generated by the linearly independent subsets of $\{2\mathbf{w}_1,\ldots,2\mathbf{w}_r\}$. The existence of such a subdivision exists by Theorem 2.1. By Corollary 3.13, subdivision $\mathscr{C}$ gives rise to a disjoint decomposition of $\mathcal{Z}$ into half-open parallelepipeds of the $\Diamond$-type. In particular, the half-open parallelepipeds of the decomposition are half-open variants of the maximal cells in $\mathscr{C}$. Thus we have

$$\mathcal{Z} = \bigsqcup_{i \in [r]} \blacklozenge_i$$

and

$$\delta(\mathcal{Z}, t) = \sum_{i \in [r]} \delta(\blacklozenge_i, t)\ ,$$

where $\blacklozenge_i$ is the appropriate half-open variant of $\lozenge_i$ in $\mathscr{C}$.

For all $i \in [r]$, the edges of $\blacklozenge_i$ are lattice centrally symmetric. Therefore, by Corollary 5.2 the coefficients of $\delta(\blacklozenge_i, t)$ are alternatingly increasing for all $i \in [r]$ and so are the coefficients of $\delta(\mathcal{Z}, t)$. □

5.2 Extensions and Open Questions

We have shown that the coefficients of the δ-polynomial for the following families of lattice polytopes are alternatingly increasing:

- Half-open lattice d-parallelepipeds with $j \geq \left\lfloor \frac{d+1}{2} \right\rfloor$ non-translate facets removed;
- Half-open lattice parallelepipeds with lattice centrally symmetric edges;
- Closed lattice zonotopes with lattice centrally symmetric edges.

We are interested in what more we can say about the inequality relations on the coefficients of the δ-polynomial for the following more general families of lattice polytopes:

- Half-open lattice centrally symmetric lattice parallelepipeds;
- Half-open lattice parallelepipeds with an interior lattice point;
- Lattice centrally symmetric lattice zonotopes;
- Lattice zonotopes with an interior lattice point;
- Half-open lattice zonotopes (i.e., lattice zonotopes with non-translate facets removed).

A further extension of our work which remains open is whether we can define appropriate bivariate permutation statistics which generalize the (A, j)- and (B, ℓ)-Eulerian numbers parallel to Brenti's q-Eulerian numbers in [4]. In particular, can we define bivariate statistics and polynomials which will further generalize the classical theory of Eulerian numbers and polynomials?

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